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A theory of (relative) discounting

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Abstract

We derive a representation theorem for time preferences (on the prize-time space) which identifies a novel notion of *relative discounting* as the key ingredient. This representation covers a variety of time preference models, including the standard exponential and hyperbolic discounting models and certain non-transitive time preferences, such as the similarity-based and subadditive discounting models. Our axiomatic work thus unifies a number of seemingly disparate time preference structures, thereby providing a tractable mathematical format that allows for investigating certain economic environments without subscribing to a particular time preference model. This point is illustrated by means of an application to sequential bargaining theory.

JEL classification: D11; D90

Keywords: Time preferences; Non-transitive preferences; Hyperbolic discounting; Time consistency; Stationarity; Present bias; Bargaining

1. Introduction

A major branch of intertemporal choice theory examines how one may evaluate the trade-offs between various alternatives that are obtained at different times. The canonical model in this regard is, of course, the so-called *exponential discounting model*. [1] Owing in part to its simplicity and in part to its built-in time consistency, this model is adopted in economics almost exclusively. Over the

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[1] The exponential discounting model was first formulated by Samuelson [31]. The axiomatic foundations of this model is explored by the definitive works of [15], and [16] in the case of time preferences over consumption streams, and of [8] in the case of time preferences over the prize-time space.

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last two decades, however, there has been an immense amount of experimental work, carried out both by economists and psychologists, which suggests strongly that the inherent time preferences of individuals are time-inconsistent, and hence depart from the exponential discounting model. [2] This has led to the consideration of various alternative models, such as the so-called *hyperbolic discounting model*. This model keeps the basic structure of the exponential discounting model, but takes the per-period discount rate as a decreasing function of time.

However, despite its incipient popularity, it seems rather premature to conclude that the experimental regularities point, unequivocally, in the direction of the hyperbolic discounting model. In fact, various other time preference models have recently been proposed in the literature, which depart from this model in radical ways, and yet are consistent with most of the experimental regularities. Two major examples in this regard are the *subadditive time preference model* of Read [28] and the *similarity-induced time preferences* of Rubinstein [30]. Like hyperbolic discounting, these alternative models incorporate certain behavioral attributes that are ignored by the exponential discounting model, such as present bias, non-stationarity, etc. Yet, unlike hyperbolic discounting, they point to the fact that the intertemporal nature of time preferences may well lead to certain types of preference cycles. For instance, both of these models allow for a situation like

$$(\$100, \text{today}) \succ (\$250, \text{two weeks from now}) \succ (\$200, \text{next week}) \succ (\$100, \text{today}),$$

where $\succ$ stands for the strict preference of the individual (who can commit to a choice today). A moment's reflection shows that there is really nothing "too odd" with this situation—the individual in question views a premium of \$50 adequate for a delay of one week, but deems a compensation of \$150 too low for a delay of two weeks. [3] Similarly, in a face-to-face bargaining situation, one may view the time delay between two consecutive offers as completely insignificant, and act as if she is indifferent between two offers like $(\$10, t)$ and $(\$10, t+1)$ for any offer period t. Yet this individual would not be indifferent between the offers $(\$10, t)$ and $(\$10, t+\tau)$ for large τ, unless she is completely insensitive to time delay. All in all, it seems clear that transitivity of time preferences, which is a precondition for the hyperbolic discounting model, is not really an innocuous assumption, especially from a descriptive angle. [4]

Furthermore, because most intertemporal choice problems are sequential in nature, non-transitive time preferences do not necessarily lead to lack of predictive power. The example above provides a case in point. Due to the non-transitivity of her preferences, there is no best alternative for the individual in the feasible set that consists of the three alternatives given above—we are apparently left with no prediction concerning her choice behavior. This is indeed the most common argument against non-transitive preferences in economic theory, but one that readily applies to the present framework *only when* the individual must commit to a choice today. Under the equally interesting scenario in which she cannot commit to a choice today, the difficulty disappears, for then, she would realize that her actual choice problem is the sequential one portrayed in Fig. 1, for which there is a unique time-consistent choice. Indeed, being rational, she knows that waiting

[2] See [9] for a brilliant survey that documents these anomalies, and provides a detailed examination of the time discounting models proposed in the literature to cope with them.

[3] For some experimental evidence about the presence of such preference cycles, see [28–30], among others.

[4] Surprisingly, there are almost no studies on *non-transitive time preferences*,despite the fact that this property arises rather naturally in the context of intertemporal choice. One exception is the work of [29] who provide empirical evidence in favor of the non-transitivity of time preferences. Another notable exception is supplied by Manzini and Mariotti [23], who propose to model time preferences as interval orders, and allow for non-transitivities to arise even within a fixed time period.

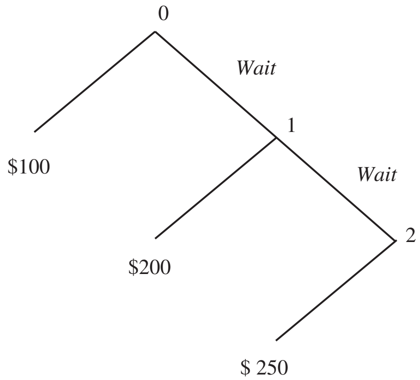

Fig. 1.

one week would actually lead to ending up with $250 with a delay of two weeks (assuming her commitment-preferences do not change in one week), and on the basis her commitment preferences, therefore, she would settle her choice problem by choosing ($100, today). Consequently, due to the sequential nature of intertemporal choice problems, non-transitivities that may arise in the time preferences of an individual with commitment need not lead to unpredictability of choice behavior.

These elementary observations suggest strongly that disallowing at the outset for preference cycles (that arise due to the passage of time) may result in considerable loss of generality. This point is only accentuated by the fact that some major alternatives of the hyperbolic discounting model portraits non-transitive time preferences (with commitment). It thus seems worth exploring time preferences at large along the lines of Koopmans [15] and Fishburn and Rubinstein [8], albeit allowing for cycles that are induced by the passage of time. The search should be guided toward developing a model general enough to admit, as special cases, the time preference models of Read and Rubinstein, along with all the standard hyperbolic discounting models proposed in the literature. Of course, to be useful, this model should not be "too general" in the vague sense that it should possess strong enough of a structure that could be taken to applications. The main objective of this paper is to develop such a model axiomatically.

We begin with a major simplification: we consider here only those time preferences that are defined on the prize-time space. Consequently, our work falls short of saying anything about time preferences over consumption streams. It rather parallels the work of Fishburn and Rubinstein [8], and is suitable for applications to bargaining and/or certain timing games. Objects of choice in the analysis are tuples like (x, t), where x corresponds to an (undated) outcome (say, monetary rewards), and t to the receival time (Section 2.1). We consider time preferences $\succsim$ over such prize-time tuples that are decreasing in time delay (the so-called "positive time preferences"). The axioms imposed on $\succsim$ are easily interpretable and are indeed satisfied by a great majority of time preference models considered in the literature (Section 2.3). They provide us with the following representation for $\succsim$:

$$(x, t) \succsim (y, s) \quad \text{if and only if} \quad U(x) \geqslant \eta(s, t) U(y), \tag{1}$$

where U is a real function on the prize space, and η is a real function on date tuples, each satisfying certain properties that will be explored later (Section 3.1). The novelty of this representation lies in its provision for the separation of "prize" and "time" aspects of prize-time tuples. Here

U corresponds to the instantaneous utility function of the individual, and $\eta(s,t)$—dubbed ∞ the "relative discount factor"—captures the significance of time delay. As we shall see, this representation allows for preference cycles to arise, but *only due to the passage of time*. Indeed, the properties of η ensure that the *static* preferences entailed by (1) are transitive. (In fact, (1) allows for a preference cycle *only if* this cycle occurs through three or more periods.)

It is important to note that representation (1) is not couched in terms of a *present value* calculation. Instead, it considers discounting as a potentially *relative* matter—hence the title of the present paper. This model interprets $(x,t) \succ (y,s)$ as follows:

> The worth at time t of the utility of y that is to be obtained at time s is strictly less than the worth at time t of the utility of x that is to be obtained at time t.

This relative discounting notion is the main novelty of our representation—instead of looking for the values of (x,t) and (y,s) at time 0, we make the comparison at time t (or at time s).

We show below that all of the time preference models mentioned above can be represented precisely as in (1) for suitable choices of U and η. Our axiomatic analysis thus identifies certain properties common to all such time preference models. Moreover, the representation seems tight enough to be useful not only for economic applications, but also for experimental testing. This is important, for the large number of experimental studies that focus on the "estimation" of discount rates assume particular parametric functional forms, and thus are vulnerable to functional misspecification errors. Having a general functional representation (that includes several rival theories within) may, then, help improving the accuracy of empirical work on time preferences.

Perhaps more importantly, representation (1) is quite "tractable." Even though it is more general than the models of Read [28] and Rubinstein [30], conducting economic analysis with this model is significantly easier than with those models. For instance, we show below that, unlike the models of Read and Rubinstein, it is very easy to identify under what conditions on U and η this model would entail behavior like transitivity, non-stationarity, present bias, etc. (Sections 3.2–4) and it generalizes easily to the context of multidimensional prize spaces (Section 4.1). In fact, despite its generality, the present model is tight enough to be taken directly to applications. Especially when coupled with the time consistency hypothesis, it allows one to study many interesting economic problems without subscribing to a particular time preference model. This point is demonstrated in Section 3.7 by means of an application to the classical Rubinstein bargaining theory. The proofs of all formal results of the paper appear in the Appendix.

2. Time preferences

2.1. Main definition

Let X be a non-empty set, and interpret this set as an (undated) outcome space. We model time continuously, and distinguish the members of X from each other according to when they are received. The preferences of the decision maker are, then, defined over the space

$$\mathcal{X} := X \times [0, \infty).$$

(As usual, we metrize $[0,\infty)$ by the Euclidean metric, and when X is itself a metric space, we metrize $\mathcal{X}$ by the product metric.) An element (x,t) of $\mathcal{X}$ is interpreted as the situation in which the agent receives the outcome x in period t.

Formally speaking, then, the time preference of an individual in this paper is modeled by means of a binary relation $\succsim$ on $\mathcal{X}$. We interpret the statement $(x,t) \succsim (y,s)$ as meaning that the individual prefers receiving x at time t to receiving y at time s in the sense that, *if she could commit in period*

0 *to one of these*, *she would choose to commit to consume x at date t*. Consequently, one should view $\succsim$ as modeling the *commitment-preferences* of the agent.

In what follows, the symmetric part of any binary relation $\succsim$ on $\mathcal{X}$ is denoted by $\sim$, and the asymmetric part of it by $\succ$. Moreover, for each $t \in [0, \infty)$, by the ***t*th time projection** of $\succsim$, we mean the binary relation $\succsim_t$ on X defined as $x \succsim_t y$ iff $(x, t) \succsim (y, t)$. Consequently, we think of $\succsim_0$ as the preference relation of the individual over (undated) outcomes *at present*.

The following definition introduces the main objects of the present analysis.

Definition. For any metric space X, a binary relation $\succsim$ on $\mathcal{X}$ is said to be a **time preference** on $\mathcal{X}$ if it satisfies the following conditions:

(i) $\succsim$ is complete and continuous, [5]
(ii) $\succsim_0$ is complete and transitive,
(iii) $\succsim_0 = \succsim_t$ for each $t \geqslant 0$.

The completeness and continuity of a time preference $\succsim$ are, of course, standard requirements. While we do not require $\succsim$ to be transitive, we wish to allow for preference cycles that may arise solely due to the passage of time, so the material preferences of the individual at time 0 are assumed to be transitive. The final requirement here is that the material tastes of the agent are *unchanging* through time.

The potential non-transitivity of time preferences forces our axiomatic analysis to be of a different flavor than the related literature, mainly because our definition leaves room for time preferences that cannot be represented by utility functions defined on $\mathcal{X}$. Non-transitive preferences are, of course, studied elsewhere in decision theory; see Fishburn [6] for a survey. Indeed, a satisfactory theory of representation exists for certain non-transitive preference relations the strict parts of which are transitive, namely for interval orders and/or semiorders. There are also some works that study more general representations over product sets (cf. [3,5,34]), where each term of the product set is treated symmetrically, but unfortunately, the axiomatic bases of the representation theorems of this literature lack, at large, an intuitive grounding (more on this in Section 4.2). At any rate, we cannot make use of either of these approaches here.

Indeed, we need to allow for even the strict part of a time preference to be non-transitive (recall the example given in Section 1). Moreover, because we wish to allow for preference cycles that arise *only due to the passage of time*, our model, per force, treats the outcome space X and the time space $[0, \infty)$ *asymmetrically*. For example, a time preference $\succsim$ on $\mathcal{X}$ permits the occurrence of a cycle like $(x, t) \succ (x, t + 2) \sim (x, t + 1) \sim (x, t)$. Indeed, with the context of bargaining theory in mind, we surely wish to allow for such cycles. However, our definition of time preference does not permit a cycle like $(x_0, t) \succ (x_2, t) \sim (x_1, t) \sim (x_0, t)$. For, such a cycle would mean that the static preferences of the agent are non-transitive, violating the precondition that the non-transivities arise only due to the passage of time. In fact, in the context of monetary prizes, such cycles would conflict with the monotonicity of one's preferences for money. [6]

[5] For any metric space A, we say that a binary relation R on A is **continuous** if it has the closed graph property, that is, if $\lim a_m \; R \; \lim b_m$ holds for all convergent sequences (a_m) and (b_m) in A such that $a_m \; R \; b_m$ for all m.

[6] The differential treatment of time and prizes is, of course, natural. Not only is that time is not a commodity to be consumed, it is altogether a different object than a prize, insofar as its primitive formulation is concerned. In particular, it is imperative that a time space has, in general, an additive semigroup structure. (Otherwise, for instance, the notion of "stationarity" would be ill-defined.) By contrast, a prize space obviously need not possess such a structure in general.

In passing, we should note that in experimental studies and in many applications of the theory of intertemporal choice, one works with a model in which time is taken to be a *discrete* variable. Unfortunately, adopting the discrete time framework in the present approach leads to some technical difficulties, which are best avoided at this stage. We thus work with time preferences on $\mathcal{X}$ in this paper.[7] In applications, one can always take the projection of these preferences to $X \times \{0, 1, \ldots, T\}$, with T being finite, or to $X \times \mathbb{Z}_+$.

2.2. Axioms for time preferences

Throughout this subsection $\succsim$ stands for a binary relation on $X \times [0, \infty)$, where X is any non-empty set. All axioms are imposed on $\succsim$.

(**A1**) (Time sensitivity) *For any* $x, y \in X$ *and* $t \geqslant 0$, *there exists an* $s \geqslant 0$ *such that* $(x, t) \succsim (y, s)$.

This axiom says simply that any (feasible) amount of prizes would be undesirable if it is to be obtained sufficiently late. For example, if $X = (0, \infty)$, (A1) says that, one would prefer receiving \$1 at some date t to getting \$100 (or \$1000, or whatever) at some sufficiently later date s. In effect, (A1) says that $\succsim$ does not ever become insensitive to time delay.

While (A1) conditions $\succsim$ to never become insensitive to time, it allows for it to get insensitive to outcomes. For instance, (A1) allows an individual to view a given time delay, say 10 time periods (or 100, or whatever) as so excessive that no feasible amount of a monetary gain can convince her to wait this long. In most applications, however, one does not impose any upper bound on monetary gains, and in that case all standard models (in which the utility for money is taken as an unbounded function) disallow this sort of an insensitivity to monetary outcomes. We do so as well by means of our next axiom.

(**A2**) (Outcome sensitivity) *For any* $x \in X$ *and* $s, t \geqslant 0$, *there exist* $y, z \in X \backslash \{x\}$ *such that* $(z, s) \succsim (x, t) \succsim (y, s)$.

Roughly speaking, this axiom means that "delay" can always be compensated by outcomes. For example, if $X = (0, \infty)$ and $s > t = 0$, then (A2) says that the agent prefers getting a large "enough" sum of money z at date s to receiving x dollars today, but prefers getting x now to receiving a small "enough" sum of money $y \neq x$ at date s. (Of course, what is "enough" depends on the size of x and the delay s.)

(**A3**) (Monotonicity) *For any* $x, y, z \in X$ *and* $s, t, r \geqslant 0$, *if* $r \leqslant t$ *and* $z \succsim_0 x$, *then*

$$(x, t) \succsim (y, s) \quad \textit{implies } (z, r) \succsim (y, s).$$

Moreover, if either the antecedent $\succsim$ *or* $\succsim_0$ *holds strictly, then* $(z, r) \succ (y, s)$.

This property is an ordinal formulation of the idea that $\succsim$ is "decreasing" in time and "increasing" in prizes. (Recall that $\succsim_0$ is the present material preferences.) It is thus unexceptionable for a theory of *positive* time preferences in which, by definition, delay is undesirable and prizes desirable.

Our next axiom can be viewed as a separability property which ensures that the disutility of time delay are independent of the size of outcomes.

(**A4**) (Separability) *For any* $x, y, z, w \in X$ *and* $s_1, s_2, t_1, t_2 \geqslant 0$, *if*

$$\begin{array}{l} (x, t_1) \sim (y, s_1) \\ \qquad\qquad \textit{and } (x, t_2) \sim (y, s_2) \quad \textit{then } (z, t_2) \sim (w, s_2). \\ (z, t_1) \sim (w, s_1). \end{array} \tag{2}$$

[7] Discrete-time formulation and derivation of all of our results are available from the authors upon request.

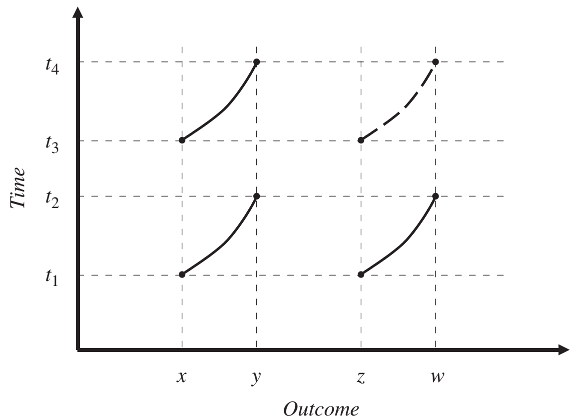

Fig. 2.

This is a separability condition analogues of which are widely used in certain branches of decision theory.[8] To illustrate its basic content, let $X = (0, \infty)$ and suppose the decision maker is indifferent between receiving x today ($t_1 = 0$) and $y > x$ tomorrow ($s_1 = 1$), and similarly between receiving z today and $w > z$ tomorrow. Then $y - x$ may be thought of as the delay premium for postponing the reward x today for one period, and similarly for $w - z$. Now suppose we also know that $y - x$ is the delay premium for waiting $s_2 - t_2$ periods instead of receiving x at time t_2. If the agent's evaluation of "time" is independent of the prizes involved in the comparison, then, exactly $w - z$ should be the delay premium for waiting $s_2 - t_2$ periods instead of getting z at time t_2. Property (A4) ensures precisely this to take place, that is, it separates the premium for delay from the particular reward (Fig. 2).[9] The structural strength (A4) is somewhat curtailed in the present context where non-transitivity of intertemporal preferences is allowed. To be able to fully separate the evaluation of time and prizes, we need an additional independence requirement here (see Theorem 3). This constitutes our final axiom.

(**A5**) (Path independence) *For any* $x, y, z, w \in X$ *and* $t_1, t_2, t_3 \geqslant 0$, *if*

$$\begin{array}{l} (x, t_1) \sim (y, t_2) \\ \qquad\qquad \textit{and } (y, t_2) \sim (w, t_3) \quad \textit{then } (x, t_2) \sim (z, t_3). \\ (z, t_1) \sim (w, t_2). \end{array} \tag{3}$$

Like (A4), property (A5) too is satisfied by virtually all time preference models used in the literature (Fig. 3).[10] To understand the structure that this axiom introduces into the picture, let us again take $X = (0, \infty)$, and fix any $t_1 < t_2 < t_3$. Now consider an individual who declares $(a, t_1) \sim (b, t_2)$ and $(c, t_2) \sim (d, t_3)$. Then, evidently, the *total* delay premium of postponing a at time t_1 by $t_2 - t_1$ periods *and* postponing c at time t_2 by $t_3 - t_2$ periods is $(b - a) + (d - c)$ for this individual. Applying this notion to the antecedents of (A5), therefore, we see that the total

[8] In the theory of additive utility representation this property is called either the *Reidemaister condition* [4,17,35] or the *corresponding tradeoffs condition* [13]. It is also commonly used in non-expected utility theory [1,36].

[9] Fig. 2 provides a visual illustration of (A4): the presence of the solid indifference curves (i.e. the antecedent in (2)) implies the presence of the dashed indifference curve (i.e. the consequent in (2)).

[10] Fig. 3, whose interpretation is identical to that of Fig. 2, provides a visual illustration of (A5).

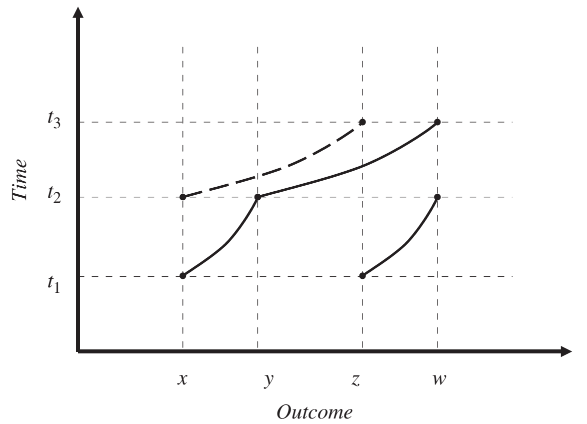

Fig. 3.

premium entailed by $(x, t_1) \sim (y, t_2)$ and $(y, t_2) \sim (w, t_3)$ is $(y-x)+(w-y) = w-x$, and that entailed by $(z, t_1) \sim (w, t_2)$ and $(x, t_2) \sim (?, t_3)$ is $(w-z) + (?-x)$. Clearly, the two aggregate premia are the same iff $? = z$, that is, $(x, t_2) \sim (z, t_3)$, as is warranted by (A5). In this sense (A5) corresponds to a *path independence* property which requires the aggregate premium for delaying rewards be independent of the order in which the premia are extracted.

Perhaps another way looking at this path independence property may be worth mentioning here. Let us agree to say that b is the "price" of postponing the alternative a at time t by $s-t$ periods if $(a, t) \sim (b, s)$, and write $b = \text{price}_{t \to s}(a)$. Then, assuming (for illustrative purposes) that the transitivity of $\succsim$, the antecedents of (A5) may be viewed as saying $\text{price}_{t_1 \to t_2}(z) = \text{price}_{t_1 \to t_3}(x)$, both of these numbers being equal to w. In turn, (A5) requires a tight connection between x and z, namely, it demands z be the price of postponing x by $t_3 - t_2$ periods. Therefore, for transitive $\succsim$, (A5) corresponds to none other than the following path independence property:

$$\text{price}_{t_1 \to t_2}(\text{price}_{t_2 \to t_3}(x)) = \text{price}_{t_1 \to t_3}(x)$$

or, equivalently, $\text{price}_{t_2 \to t_3}(x) = \text{price}_{t_2 \to t_1}(\text{price}_{t_1 \to t_3}(x))$. This property is, of course, intrinsic to ranking dated outcomes on the basis of present value computations. As we shall show below, it is consistent with many other methods as well.

As for examples, we note that any discounting model which envisages that "outcomes" and "time" are separable in such a manner that all outcomes are discounted the same way (such as the exponential and/or hyperbolic discounting models, among others) would trivially satisfy properties (A4) and (A5).[11] Moreover, both of these properties enjoy fairly reasonable interpretations, which at some level, resemble present value computations. It thus seems warranted to explore the implications of (A4) and (A5) for the theory of time preferences. Yet (A5) is a more complicated axiom than (A4), and its appeal is not unquestionable. For this reason, we will explore in Section 4.2 how exactly the theory that emanates from the properties considered in this section would be altered by the absence of (A5).

[11] Conversely, time preferences that exhibit the so-called *magnitude effect* (the decrease of the rate of discounting with the utility value of the outcomes) would in general violate both (A4) and (A5).

3. Time preferences over monetary outcomes

3.1. Main results

Our main objective here is to give a complete characterization of the class of time preferences that satisfy the axioms introduced above, thereby unifying a great variety of time preference models considered in the literature. In this section we will provide such a characterization in the case where the prizes are given in monetary terms in the sense that X is a non-empty interval in $\mathbb{R}$. (A significant generalization is provided in Section 4.1.) The main advantage of this case is that it warrants the following innocuous assumption which says that an agent strictly prefers the larger prize to a smaller one, provided that both prizes are to be received today.

(**A6**) (Monotonicity in prizes) $\succsim_0$ *is strictly increasing on* X.

The following is our first main result.

Theorem 1. *Let X be an open interval in $\mathbb{R}$ and $\succsim$ a binary relation on $\mathcal{X}$. $\succsim$ is a time preference on $\mathcal{X}$ that satisfies properties* (A1)–(A6) *if, and only if, there exist an increasing homeomorphism $U : X \to \mathbb{R}_{++}$ and a continuous map $\eta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ such that, for all $x, y \in X$ and $s, t \geqslant 0$,*

$$(x, t) \succsim (y, s) \quad \textit{iff} \ U(x) \geqslant \eta(s, t)U(y), \tag{4}$$

while (*i*) *$\eta(\cdot, t)$ is decreasing with $\eta(\infty, t) = 0$, and* (*ii*) *$\eta(s, t) = 1/\eta(t, s)$.*

Theorem 1 provides a representation for time preferences over monetary outcomes by means of *separating* the "delay" and "prize" aspects. In this representation U simply serves as a utility function for undated outcomes; we have $(x, t) \succsim (y, t)$ iff $U(x) \geqslant U(y)$ for all $x, y \in X$ and $t \geqslant 0$ (since, by (ii), $\eta(t, t) = 1$ for each t). In turn, η captures the importance of time for the individual in question. The model is one of positive time preference; for any $x \in X$, we have $(x, t) \succsim (x, s)$ iff $t \geqslant s \geqslant 0$ (since, by (i) and (ii), $0 < \eta(s, t) \leqslant 1$ iff $t \leqslant s$).

More generally, the representation tells us how the agent would "aggregate" the time and outcome aspects when comparing two dated outcomes (x, t) and (y, s). For instance, if $U(x) \geqslant U(y)$ and $t > s$, the agent has to decide whether she should receive the better outcome later, or the worse outcome sooner. The representation maintains that this is done by comparing the outcome value $U(x)$ with the outcome value $U(y)$ that is discounted according to the relative receival dates of x and y. Thus, $\eta(s, t)$ acts as a *relative discount factor* which tells us exactly how the agent disvalues the time delay that takes between periods s and t. If $U(x) > \eta(s, t)U(y)$, the time difference does not compensate for the value difference in outcomes, so she goes for (x, t) as opposed to (y, s). If $U(x) < \eta(s, t)U(y)$, then the time aspect of the comparison outweighs the prize aspect, and she chooses (y, s) over (x, t). The case $U(x) = \eta(s, t)U(y)$ corresponds to the case of indifference.

In the standard discounted utility model, one discounts the utility of a given alternative x to be received at time t (according to some discount function), and compares the resulting value to similarly discounted utilities of other outcomes received at possibly other dates. This corresponds to a notion of *absolute* discounting in the sense that we may then compare all prize-time tuples (x, t) by using their present values. By contrast, the representation obtained in Theorem 1 says that we cannot in general talk about "the" present value of a dated outcome; discounting is a potentially *relative* matter. That is, the utility of x can be discounted only relative to its receival date *and* some other receival date t. Put differently, $(x, t) \succ (y, s)$ should be interpreted as follows: The worth at time t of the utility of y that is to be obtained at time s (i.e. $U(y)\eta(s,t)$) is strictly less than the worth at time t of the utility of x that is to be obtained at time t (i.e. $U(x) = U(x)\eta(t,t)$). This relative discounting notion is the main novelty of our representation; instead of looking for the values of (x,t) and (y,s) at time 0, we make the comparison at time t (or at time s; by property (ii), this does not matter).

Going back to the evaluation of Theorem 1, we note that the outcome-utility function U found here is not only continuous and strictly increasing, but it is also unbounded. While the latter requirement is not innocuous, it hardly seems unacceptable; it is a fair price to pay for assuming the outcome sensitivity property (A2) (which is, in turn, essential for the "only if" part of Theorem 1). Conditions (i) and (ii) are, on the other hand, easily interpreted. Condition (i) simply says that the agent prefers getting an outcome earlier than later, and that the value of any given outcome is negligible, provided that this outcome is to be obtained sufficiently late. Condition (ii) ensures that "time delay" is evaluated symmetrically back and forth through time. For instance, if $s > t$, then the utility of an outcome that is received at s instead of t is discounted by $\eta(s,t) \leqslant 1$, and conversely, the utility of an outcome that is received at t instead of s is inversely discounted by $1/\eta(s,t) \geqslant 1$.

Insofar as the uniqueness related matters are concerned, one can show that the utility function U and the relative discount factor η found in Theorem 1 are unique up to *simultaneous* positive exponential transformations. To state this result in precise terms, we introduce the following bit of terminology that we shall use later as well.

Definition. Let X be a metric space. We say that a time preference on $\mathcal{X}$ is **represented by** (U, η) if $U : X \to \mathbb{R}_{++}$ is an homeomorphism and $\eta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ is a continuous map such that (4) and conditions (i) and (ii) of Theorem 1 hold for all $x, y \in X$ and $s, t \geqslant 0$.

The following result complements Theorem 1.

Proposition 1. *Let X be a metric space, and $\succsim$ a time preference on $\mathcal{X}$ which is represented by (U, η). Then $\succsim$ is represented by (V, ζ) if and only if $V = bU^a$ and $\zeta = \eta^a$ for some $a, b > 0$.*

In the following subsections, we consider a few refinements of the representation given in Theorem 1 by imposing on $\succsim$ some additional properties such as transitivity, stationarity and present bias. We then consider some non-transitive time preference models that are covered by Theorem 1, and explore how one may include in the model a time-neutral outcome (such as the disagreement outcome in the Rubinstein bargaining model).

3.2. Transitive time preferences over monetary outcomes

As noted above, representation (4) allows for non-transitive time preferences; we will consider concrete examples below. In fact, it is easily shown that a time preference that is represented as in Theorem 1 is transitive if and only if

$$\eta(t,r) = \eta(t,s)\eta(s,r) \quad \text{for all } r, s, t \geqslant 0. \tag{5}$$

This observation leads to a complete characterization of transitive time preferences that satisfy (A1)–(A6). It turns out that, under the transitivity hypothesis, the representation becomes one of *absolute discounting*, that is, comparison of dated outcomes takes place through a present value computation.

Corollary 1. *Let X be an open interval in $\mathbb{R}$ and $\succsim$ a binary relation on $\mathcal{X}$. $\succsim$ is a transitive time preference on $\mathcal{X}$ that satisfies properties* (A1)–(A6) *if, and only if, there exist an increasing homeomorphism $U : X \to \mathbb{R}_{++}$ and a decreasing and continuous map $D : \mathbb{R}_+ \to (0, 1]$ such that $D(0) = 1$, $D(\infty) = 0$, and*

$$(x, t) \succsim (y, s) \quad \textit{iff} \;\; D(t)U(x) \geqslant D(s)U(y) \tag{6}$$

for all $(x, t), (y, s) \in \mathcal{X}$.

While it is illuminating to see the potential contribution of transitivity to Theorem 1, Corollary 1 is not really a new finding. It parallels closely the characterization of multiplicative discounting model given by Fishburn and Rubinstein [8]. Conceptually speaking, however, it is a useful observation, for it shows that all standard models of time preferences over monetary gains are covered by Theorem 1. To see this more clearly, let $X := (0, \infty)$, and let U be an arbitrary strictly increasing and continuous real function on X with $U(0+) = 0$ and $U(\infty) = \infty$. The *exponential discounting model* defines a time preference on $\mathcal{X}$ by $(x, t) \succsim (y, s)$ iff $\delta^t U(x) \geqslant \delta^s U(y)$, where $\delta \in (0, 1)$ is the discount factor. Obviously, such a time preference belongs to the class characterized by Corollary 1 (where $D(t) = \delta^t$ for all $t \geqslant 0$) and hence by Theorem 1 (where $U = u$ and $\eta(s, t) = \delta^{s-t}$ for all $s, t \geqslant 0$). Similarly, any commitment-preference of the *hyperbolic discounting model* is captured by Corollary 1, and hence by Theorem 1. [12]

3.3. Stationary time preferences over monetary outcomes

One of the main features of the classical exponential discounting model is that of its stationarity. That is, in that model, the effect of time enters into the comparison of two dated outcomes only through the *difference* between the receival times of these outcomes. In the present framework, this property is defined as follows.

Definition. Let X be a non-empty set. A time preference $\succsim$ on $\mathcal{X}$ is said to be **stationary** if

$$(x, t) \succ (y, s) \quad \text{iff } (x, t + \tau) \succ (y, s + \tau)$$

for all $(x, t), (y, s) \in \mathcal{X}$ and $\tau \in \mathbb{R}$ such that $s + \tau, t + \tau \geqslant 0$.

The following immediate corollary of Theorem 1 demonstrates the structure of the stationary relative discounting in the present setup.

Corollary 2. *Let X be an open interval in $\mathbb{R}$ and $\succsim$ a binary relation on $\mathcal{X}$. $\succsim$ is a stationary time preference on $\mathcal{X}$ that satisfies properties* (A1)–(A6) *if, and only if, there exist an increasing homeomorphism $U : X \to \mathbb{R}_{++}$ and a decreasing and continuous map $\zeta : \mathbb{R} \to \mathbb{R}_{++}$ such that $\zeta(\infty) = 0$, $\zeta(a) = 1/\zeta(-a)$ for all $a \geqslant 0$, and*

$$(x, t) \succsim (y, s) \quad \textit{iff} \;\; U(x) \geqslant \zeta(s - t)U(y) \tag{7}$$

for all $(x, t), (y, s) \in \mathcal{X}$.

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[12] Perhaps the most popular such example has it that $D(t) = 1/(1 + t)$ for all $t \geqslant 0$ in the language of Corollary 1, and $\eta(s, t) = (1 + t)/(1 + s)$ for all $s, t \geqslant 0$ in the language of Theorem 1.
We note that the representation of Theorem 1does not capture the so-called *quasi-hyperbolic discounting model* of [26,19] due to a technicality. The discount function of that model is not continuous at 0 (for $D(0) = 1 > D(0+)$). If one wishes to adopt our model within a discrete framework, however, that model too becomes a special case of the model characterized by Corollary 1 (See Fig. 4).

The next result shows that the only stationary and transitive time preference model that belongs to the class characterized by Theorem 1 is that of the exponential discounting model. Thus the axiomatic framework that underlies the representation given in Theorem 1 is a relatively tight one in that disallowing for non-transitivity and non-stationarity in this framework amounts to exponential discounting.

Corollary 3. *Let X be an open interval in $\mathbb{R}$ and $\succsim$ a binary relation on $\mathcal{X}$. $\succsim$ is a stationary and transitive time preference on $\mathcal{X}$ that satisfies properties* (A1)–(A6) *if, and only if, there exist an increasing homeomorphism $U : X \to \mathbb{R}_{++}$ and a $\delta \in (0, 1)$ such that*

$$(x, t) \succsim (y, s) \quad \textit{iff } \delta^t U(x) \geqslant \delta^s U(y)$$

for all $(x, t), (y, s) \in \mathcal{X}$.

While it is illuminating to see how the exponential discounting model is embedded in the class of time preferences characterized by Theorem 1, this result, like Corollary 1, should not be considered as a new finding. Indeed, apart from a few technical details, Corollary 3 can be thought of as a special case of the well-known characterization of the exponential discounting model by Fishburn and Rubinstein [8].

3.4. Present bias

In view of the ample experimental evidence that shows that most individuals attach a special significance to receiving an outcome "today" (that is, at date 0) as opposed to some later date, it is of interest to identify which sorts of time preferences that we consider here exhibit such a present bias. We first need to formulate the idea of "present bias" within our ordinal framework.

Definition. For any non-empty set X, a time preference $\succsim$ on $\mathcal{X}$ is said to have **present bias** if

$$(x, t) \succsim (y, s) \quad \text{implies } (x, 0) \succsim (y, s - t),$$

for all $x, y \in X$, and $s > t \geqslant 0$. It is said to have **strong present bias** if it has present bias, and if, for any $s > t > 0$, there exist $x, y \in X$ such that

$$(x, t) \sim (y, s) \quad \text{and} \quad (x, 0) \succ (y, s - t). \tag{8}$$

The interpretation of this definition is straightforward. If $(x, t) \succsim (y, s)$, and $s > t$, then we understand that the delay that occurs between t and s is not compensated by the relative magnitude of the utility of y over that of x. If the agent has present bias, then it should certainly be the case that she prefers getting x *now* as opposed to receiving y after waiting for a period of length $s - t > 0$. Strong present bias is similarly interpreted.

The following result identifies the structure of time preferences that belong to the class characterized by Theorem 1 and that have (strong) present bias.

Corollary 4. *Let X be an open interval in $\mathbb{R}$ and $\succsim$ a binary relation on $\mathcal{X}$. $\succsim$ is a time preference on $\mathcal{X}$ that satisfies properties* (A1)–(A6) *and has (strong) present bias if, and only if, $\succsim$ is represented by some (U, η) such that $\eta(s, t)(>) \geqslant \eta(s - t, 0)$ holds whenever $s > t > 0$.*

As for examples, we note that any stationary time preference has present bias, but not strong present bias. For instance, the exponential discounting model exhibits present bias, but not strong present bias. By Corollary 4, we also find that any time preference that is represented as in Corollary 1 has (strong) present bias if and only if $D(s)/D(t)(>)\geqslant D(s-t)$ whenever $s>t>0$.

3.5. *Non-transitive time preferences: examples*

The time preferences that we considered in our examples so far have all been transitive. We have not yet considered any examples of relative discounting models that portrait non-transitive preference. We examine next a few concrete examples of such preferences under the guideline of Theorem 1.

Example 1 (*A non-transitive stationary time preference*). Let $X:=(0,\infty)$, define η on $\mathbb{R}^2_+$ as

$$\eta(s,t):=\begin{cases}1/(s-t) & \text{if } s-t>1,\\ 1 & \text{if } 1\geqslant s-t\geqslant 0,\end{cases}$$

and $\eta(s,t):=1/\eta(t,s)$ if $t>s$ (see Fig. 4), and consider the binary relation $\succsim$ on $\mathcal{X}$ defined by $(x,t)\succsim(y,s)$ iff $x\geqslant\eta(s,t)y$. Such an individual does not discount a time period of length less than 1; she does not perceive any cost to delaying the receipt of a prize for this long a time period. However, she considers delays of more than one period costly. In particular, a delay of two time periods halves the value of the prize for her. It is readily checked that $\succsim$ is a time preference that belongs to the class characterized by Corollary 2. Yet even the strict part of $\succsim$ is not transitive. For instance, $(6,2)\succ(5,1)\succ(4,0)$ but $(4,0)\succ(6,2)$. Since $\succsim$ is stationary, it has present bias, but not strong present bias.

Example 2 (*Subadditive discounting*). In an interesting study, Read [28] observes that non-stationary phenomena like strong present bias may be explained by means of subadditive discounting (as opposed to hyperbolic discounting). *Subadditive discounting* corresponds to the idea that the total discounting over a given period may be greater than the sum of discounting when that period is divided into two parts. In the language of the present model, this means that, for any $r>s>t\geqslant 0$, if $(x,r)\sim(y,t)$ and $(x,r)\sim(z,s)\sim(w,t)$, then $w\succ_0 y$. While Read [28] discusses possible causes of subadditive discounting, and provides experimental evidence in support of its presence, he does not provide a concrete time preference model that incorporates subadditive discounting. The notion of relative discounting, however, can be readily used to obtain various specific subadditive discounting models. Indeed, if $\succsim$ is represented by (U,η), and if $\eta(t,r)>\eta(t,s)\eta(s,r)$ for any $r>s>t\geqslant 0$ (compare with (5)), then $\succsim$ corresponds to a subadditive discounting model. [13] One can also use Theorem 1 in a similar fashion to obtain hybrid models where, for instance, short time intervals are subject to additive discounting, and longer ones to subadditive discounting.

Example 3 (*The Rubinstein time preference model*). In his critique of hyperbolic discounting, Rubinstein [30] points to the fact that decision-making procedures based on similarity relations

[13] For example, if $f:\mathbb{R}_+\to\mathbb{R}_+$ is any convex and strictly increasing function with $f(0)=0$ and $f(\infty)=\infty$, and η is defined on $\mathbb{R}^2_+$ as $\eta(s,t):=\begin{cases}\exp f(|t-s|) & \text{if } s\leqslant t,\\ 1/\exp f(|t-s|) & \text{if } s>t,\end{cases}$ then the associated (stationary) time preference (given by Theorem 1) entails subadditive discounting. Fig. 4 depicts how this particular discounting model compares with those of the other time preference models considered so far.

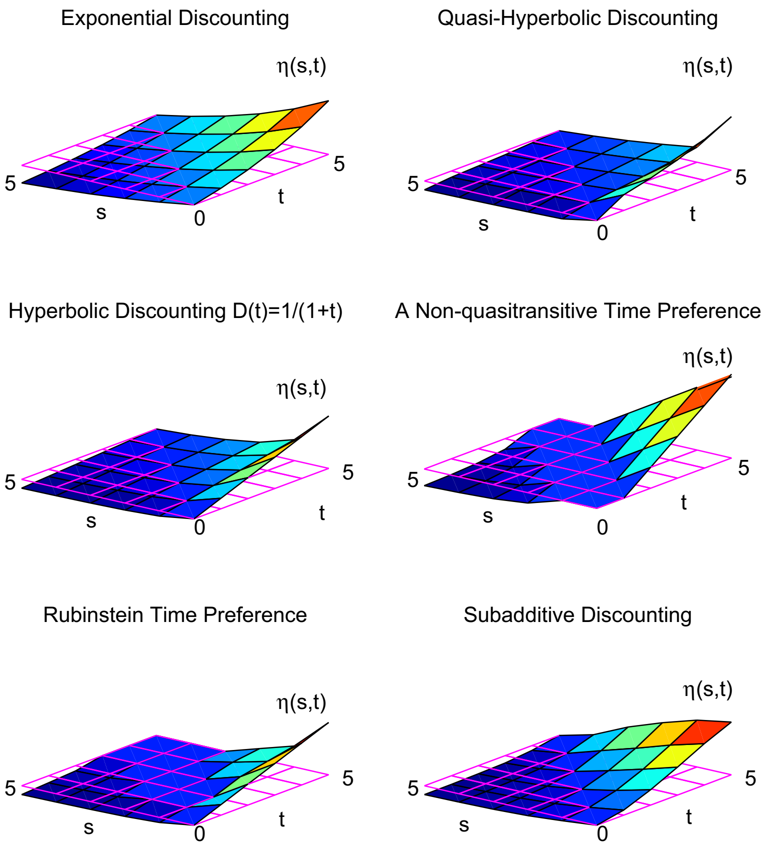

Fig. 4.

may better explain the experimental findings. One way of formalizing this suggestion in the present context is as follows.

Let $\approx_X$ and $\approx_T$ be reflexive and symmetric binary (i.e. similarity) relations on $X := (0, \infty)$ and $[0, \infty)$, respectively. (We interpret $x \approx_X y$ as the agent viewing the alternatives x and y as "similar." The expression $t \approx_T s$ is interpreted analogously.) Let $\succsim$ be a binary relation on $\mathcal{X}$, and consider the following postulates:

(R1) If $x \succ_0 y$ and $t \leqslant s$, then $(x, t) \succ (y, s)$.
(R2) If $t \approx_T s$, then $(x, t) \succsim (y, s)$ iff $x \succsim_0 y$.[14]
(R3) If $t \approx_T s$ does not hold, but $x \approx_X y$, then $(x, t) \succsim (y, s)$ iff $t \leqslant s$.

Rubinstein [30] views the above properties as a procedure in which the agent first checks if (R1) applies, then (R2) and then (R3), but leaves unspecified how the decision is made when none of these properties apply. To complete the model, we assume here that the agent uses a particular

[14] We depart slightly from the original formulation of Rubinstein here, which, in fact, leaves unspecified how the decision maker behaves when both $t \approx_T s$ and $x \approx_X y$ hold. We presume here that rewards are the decisive factor in such circumstances.

rule to aggregate the utility of outcome and the disutility of delay. Formally, we posit that there exists a continuous function $V : \mathcal{X} \to \mathbb{R}$ such that $(x,t) \succsim (y,s)$ iff $V(x,t) \geqslant V(y,s)$ for any (x,t) and (y,s) that fail to satisfy any of the antecedents of (R1)–(R3). Thus, $\succsim$ is completely identified by the list $(\succsim_0, \approx_X, \approx_T, V)$, which we will refer to as a **Rubinstein time preference model**.

It turns out that a variety of Rubinstein time preference models are indeed models of relative discounting, and hence, are captured by Theorem 1. To illustrate, let U be any increasing homeomorphism that maps X onto $\mathbb{R}_{++}$, and define $\succsim_0$ as the preference relation on X represented by U. Also assume that $\approx_X$ is any similarity relation on X such that there exists an $\varepsilon > 1$ such that $\frac{1}{\varepsilon} \leqslant \frac{U(x)}{U(y)} \leqslant \varepsilon$ whenever $x \approx_X y$. Now take any strictly decreasing $f : \mathbb{R}_+ \to \mathbb{R}_{++}$ with $f(\infty) = 0$, and fix any similarity relation $\approx_T$ on $[0, \infty)$ such that $\frac{1}{\varepsilon} \leqslant \frac{f(s)}{f(t)} \leqslant \varepsilon$ whenever $s \approx_T t$ does not hold. Finally, define $V : \mathcal{X} \to \mathbb{R}$ by $V(x,t) := f(t)U(x)$. It can be checked that the time preference $\succsim$ identified by the Rubinstein time-preference model $(\succsim_0, \approx_X, \approx_T, V)$ is represented by (U, η), where $\eta(s,t) := 1$ if $s \approx_T t$, and $\eta(s,t) := \dfrac{f(s)}{f(t)}$ otherwise. (See Fig. 4.) Depending on the structure of the time-similarity relation $\approx_T$, $\succsim$ may or may not have strong present bias.

Finally, we consider a model of time preferences which is not a relative discounting model.

Example 4 (*Time preferences with contemplation costs*). Let $X := (0, \infty)$ and $U : X \to \mathbb{R}_{++}$ be an increasing homeomorphism. A version of the time preference model considered by Benhabib et al. [2] posits that $\succsim$ is represented on $\mathcal{X}$ as follows:

$$(x,t) \succsim (y,s) \quad \text{iff } D(t)U(x) - c(t) \geqslant D(s)U(y) - c(s).$$

Here $D : \mathbb{R}_+ \to (0, 1]$ is a strictly decreasing (discount) function with $D(0) = 1$ and $D(\infty) = 0$, and $c : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function. (One may interpret $c(t)$ as the contemplation cost associated with planning about t periods ahead.) Obviously, $\succsim$ is a time preference that reduces to the model characterized in Corollary 1 when $c(t) = 0$ for all t. However, in general, $\succsim$ need not satisfy either (A4) or (A5), and hence lies outside of the class of time preferences characterized by Theorem 1.

3.6. Time preferences with a delay-neutral outcome

In many economic applications, one stipulates the presence of a time-independent "worst outcome" for the involved agents. (For instance, in the classical Rubinstein bargaining game and its variants, such an outcome serves as the disagreement point in which gains from trade are nil.) In the formalism of the present model, this outcome, let us denote it by x_*, must satisfy the following:

$$(x_*, t) \sim (x_*, s) \quad \text{and} \quad (x,t) \succ (x_*, s) \ \text{ for all } x \in X \setminus \{x_*\} \text{ and } s, t \in [0, \infty). \tag{9}$$

By contrast, the structure of time preferences characterized by Theorem 1 does not allow for the presence of such an outcome, precisely due to the time sensitivity property (A1). While this may at first seem problematic, it is, in fact, easily dealt with upon attaching x_* as the minimum element to X and asking (A1) to apply to all undated alternatives but x_*. More precisely, where $X = [x_*, x^*)$ with $x_* < x^* \leqslant \infty$, all we need is to replace (A1) with the following property.

(A1*) (Time sensitivity) *For any* $x, y \in X\backslash\{x_*\}$ *and* $t \geqslant 0$, *there exists an* $s \geqslant 0$ *such that* $(x, t) \succsim (y, s)$.

The resulting relative discounting model is outlined in the following offspring of Theorem 1.

Corollary 5. *Let* $X = [x_*, x^*)$, $x_* < x^* \leqslant \infty$, *and let* $\succsim$ *be a binary relation on* $\mathcal{X}$. $\succsim$ *is a time preference on* $\mathcal{X}$ *that satisfies properties* (A1*)–(A6) *if, and only if, there exist an increasing homeomorphism* $U : X \to \mathbb{R}_+$ *and a continuous map* $\eta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ *such that* (4) *and* (i) *and* (ii) *of Theorem* 1 *hold for all* $x, y \in X$ *and* $s, t \geqslant 0$.

The existence of a time-neutral outcome is thus consistent with the axiomatic system that is adopted in Theorem 1 (with (A1*) replacing (A1)). The only modification in the resulting representation is that this outcome is now assigned the utility value of zero (i.e. $U(x_*) = 0$) while the utility of any other alternative is strictly positive. Needless to say, this formulation is duly consistent with the way time-neutral outcomes are commonly modeled in applied work.

3.7. An application to bargaining theory

A major advantage of the hyperbolic discounting model is that, when combined with the time consistency principle, it turns in a rather tractable model of intertemporal choice which portraits a good deal of predictive power. At least in the case of certain economic problems, the same is true for the relative discounting model envisaged by Theorem 1 as well. To illustrate this point, and to show how the general time preferences studied here can be used in practice, we revisit in this section the Rubinstein bargaining model by using such time preferences.

Consider two players who are engaged through the standard alternating offers bargaining game in which time is discrete and the set A of all agreements is the set of all divisions of a unit cake: $A := \{(a, 1 - a) : 0 \leqslant a \leqslant 1\}$. We model the preferences of the players over $[0, \infty) \times [0, \infty)$ as represented by a pair (U, η) as in Corollary 5 with $x_* = 0$, $x^* = \infty$ and U being strictly increasing and strictly concave. These preferences are extended to $A \times \mathbb{Z}_+$ in the obvious way: Player 1, for instance, prefers $((a, 1 - a), t)$ to $((b, 1 - b), s)$ iff $U(a) \geqslant \eta(s, t)U(b)$. If no agreement is ever reached, each player receives the worst outcome 0. We refer to the resulting game as the **generalized Rubinstein** (U, η)**-bargaining game**, or for short, (U, η)-*bargaining game*. If $\succsim$ corresponds to an exponential time discounting model (i.e. when $\eta(s, t) = \delta^{s-t}$ for all s and t, with $0 < \delta < 1$ being the discount factor), then we refer to this game as the **standard Rubinstein** (U, δ)**-bargaining game**.

We are interested in the equilibria of the (U, η)-bargaining game that would obtain when the players determine their strategies in a *time consistent* manner. In line with the standard notion of time consistency, therefore, we treat each player at time t as a different person. [15] The preferences of these "selves" of the players are defined in the obvious way: the period t "self" of player 1 prefers $((a, 1-a), \tau)$ to $((b, 1-b), \tau')$ iff $U(a) \geqslant \eta(\tau - t, \tau' - t)U(b)$, where $\tau, \tau' \in \{t, t+1, \ldots\}$. [16] We

[15] The idea goes back to [33,26], and is adopted routinely in applications. In this sort of a formulation, time consistency is defined as (possibly a refinement) of the subgame perfect equilibria of the induced game played by the "selves" of the agent. See, for instance, [10,11,14,25].

[16] The preferences of this "self" over agreements reached *before* time t are irrelevant; for concreteness we may assume that this player receives 0 (or any a $\in [0, 1]$) if the game ends before period t is reached.

refer to any (stationary) subgame perfect equilibrium of the resulting (infinite-player) game as a **time consistent (stationary) subgame perfect equilibrium** of the (U, η)-bargaining game.[17]

Clearly, the subgame perfect equilibrium of the standard Rubinstein (U, δ)-bargaining game is the unique time consistent subgame perfect equilibrium of the (U, η)-bargaining game, where $\eta(s, t) = \delta^{s-t}$ for all $s, t \in \mathbb{Z}_+$. This fact generalizes to an arbitrary (U, η)-bargaining game with $\eta(1, 0) < 1$.

Proposition 2. *There is a unique time consistent subgame perfect equilibrium of any generalized Rubinstein* (U, η)*-bargaining game with* $\eta(1, 0) < 1$*. This equilibrium equals the unique subgame perfect equilibrium of the standard Rubinstein* $(U, \eta(1, 0))$*-bargaining game.*

Under the proviso of time consistency, therefore, the nature of a (U, η)-bargaining game with $\eta(1, 0) < 1$ is identical to the standard Rubinstein model. Due to the highly stationary nature of this game, how one models time preferences turns out to be irrelevant for the main conclusions that stem from the Rubinstein model, provided that each player discounts even a single-period *strictly*.[18]

A different scenario obtains in the case where $\eta(1, 0) = 1$, that is, when waiting for one more period at time zero is deemed costless. For concreteness, we only consider the case where $\eta(1, 0) = 1 > \eta(2, 0)$. (The generalization to the case where $\eta(1, 0) = \cdots = \eta(t, 0) = 1 > \eta(t + 1, 0)$ for some $t \in \mathbb{N}$ is straightforward.) Under this specification the time preferences of the players are necessarily non-transitive. For instance, player 1 may be indifferent between getting an outcome at time 0 and 1, and between 1 and 2, while she strictly prefers to obtain that outcome now as opposed to waiting for it for two periods. As noted in the Introduction, the plausibility of these sorts of preference cycles is in fact one of the main motivations for our analysis.

The following result complements Proposition 2.

Proposition 3. *For every* $0 \leqslant a \leqslant 1$*, there is a time consistent stationary subgame perfect equilibrium of any* (U, η)*-bargaining game with* $\eta(1, 0) = 1 > \eta(2, 0)$ *in which the outcome* $(a, 1 - a)$ *is agreed upon in either period* 0 *or* 1*. There is no stationary equilibrium of such a game which exhibits a delay of more than one period.*

To see this, fix any $0 \leqslant a \leqslant 1$, and consider the strategy profile in which any (proposing) self of player 1 always offers $(a, 1-a)$ while any (responding) self of her accepts a proposal $(b, 1-b) \in A$ iff $b \geqslant a$. Similarly, any (proposing) self of player 2 always offers $(a, 1-a)$ while any (responding) self of her accepts a proposal $(b, 1 - b) \in A$ iff $b \leqslant a$. Given that $\eta(1, 0) = 1$, this profile is easily checked to be an equilibrium in which $(a, 1 - a)$ is agreed upon in period 0.

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[17] Of course, here we identify the equilibrium action of the period t "self" of player i in the former game with the action of the player i in period t in the (U, η)-bargaining game, $i = 1, 2$.

[18] The proof of Proposition 2 follows from the standard analysis of the Rubinstein bargaining game. For any pair of agreements $(x^*, y^*) \in A^2$ such that player 1 is indifferent between $(x^*, 1)$ and $(y^*, 0)$, and player 2 is indifferent between $(x^*, 0)$ and $(y^*, 1)$, there is a subgame perfect equilibrium of the induced infinite-player game in which all "selves" of player 1 always propose x^* and accept a proposal $x \in A$ iff $U(x_1) > U(y_1^*)$, while all "selves" of player 2 always propose y^* and accept a proposal $x \in A$ iff $U(x_2) > U(x_2^*)$. Moreover, the standard Shaked-Sutton argument applies to the present setting without alteration, and hence all one needs to verify here is that there exists a unique $(x_1^*, y_1^*) \in [0, 1]^2$ such that $\eta(1, 0)U(x_1^*) = U(y_1^*)$ and $\eta(1, 0)U(1 - y_1^*) = U(1 - x_1^*)$. This is established by a standard fixed-point argument; we omit the details.

Now consider the strategy profile in which all selves of player 1 and all but period 0 selves of player 2 behave exactly as in the previous profile. The period 0 self of player 2 accepts a proposal $(b, 1-b) \in A$ iff $b < a$. This profile is an equilibrium in which $(a, 1-a)$ is agreed upon in period 1. (We leave the proof of the final assertion of Proposition 3 as an exercise.)

Many other results derived about the Rubinstein bargaining model and its numerous ramifications can be studied with the relative discounting model derived in Corollary 5. Our objective here was only to illustrate the "applicability" of this model. While it is quite general, the structural simplicity of this model enhances its usability in the context of a variety of economic problems. Adopting this model may validate certain findings that are obtained via the exponential discounting model in some applications (as in Proposition 2), and in others it may lead to new results, identifying the restrictions imposed by that model more crisply (as in Proposition 3).

4. More general models of time preferences

4.1. Characterization of time preferences over multidimensional prize spaces

The results of the previous section are all based on assumption (A6) which implies that the agent is not indifferent between any two distinct (undated) prizes. Consequently, those results are not applicable in a context in which the prize space consists of, say, commodity bundles. A useful generalization of the analysis outlined so far would obtain, then, by extending it to the case where the prize space X is a subset of an arbitrary Euclidean space. The following result provides such a generalization.

Theorem 2. *Let $n \in \mathbb{N}$, X a non-empty open box in $\mathbb{R}^n$, and $\succsim$ a binary relation on $\mathcal{X}$.*[19] *$\succsim$ is a time preference on $\mathcal{X}$ that satisfies properties* (A1)–(A6) *if, and only if, there exist continuous surjections $U : X \to \mathbb{R}_{++}$ and $\eta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ such that U is strictly increasing,* (4) *holds on $\mathcal{X}$, and*

(i) *$\eta(\cdot, t)$ is decreasing with $\eta(\infty, t) = 0$ for any $t \geqslant 0$,*
(ii) *$\eta(s, t) = 1/\eta(t, s)$ for any $s, t \geqslant 0$.*

Moreover, $V : X \to \mathbb{R}_{++}$ and $\zeta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ also satisfy these properties if and only if $V = bU^a$ and $\zeta = \eta^a$ for some $a, b > 0$.

The interpretation of this result is identical to that of Theorem 1. Moreover, all of the corollaries of Theorem 1 that we considered above are easily extended to the case covered by Theorem 2; we omit repeating the arguments.

4.2. Path-dependent time preferences

The strongest assumption that we imposed on time preferences in Theorem 1 is arguably the path-independence requirement (A5). It is thus of interest to find out how exactly this theorem would modify in the absence of this assumption. The following result provides the answer to this query.

[19] By an *open box* in $\mathbb{R}^n$, we mean the cartesian product of n open intervals (with the product topology).

Theorem 3. *Let X be an open interval in $\mathbb{R}$ and $\succsim$ a binary relation on $\mathcal{X}$. $\succsim$ is a time preference on $\mathcal{X}$ that satisfies properties* (A1)–(A4) *and* (A6) *if, and only if, there exist an increasing homeomorphism $u : X \to \mathbb{R}$ and continuous maps $F : \mathbb{R}^2 \to \mathbb{R}_{++}$ and $\eta : \mathcal{X}^2 \to \mathbb{R}_{++}$ such that, for all $x, y \in X$ and $s, t \geqslant 0$,*

$$(x,t) \succsim (y,s) \quad \textit{iff} \;\; F(u(x), u(y)) \geqslant \eta(s,t), \tag{10}$$

while:

(i) *$\eta(\cdot, t)$ is decreasing with $\eta(\infty, t) = 0$, and $\eta(s,t) = 1/\eta(t,s)$;*
(ii) *$F(\cdot, \beta)$ is strictly increasing, surjective, and $F(\alpha, \beta) = 1/F(\alpha, \beta)$ for each $\alpha, \beta \in \mathbb{R}$.*[20]

This result shows formally that the main function of (A5) is to make sure that the relative discount factor η we find in Theorem 1 can properly be interpreted as "discounting" the utility function of a given agent. Without this assumption, the agent ranks two dated outcomes (x, t) and (y, s) by separately comparing the relative utilities of x and y and the receival times s and t, and then aggregating these comparisons in an additive fashion. While this model is probably "too general" to be useful for most applied purposes—the analysis of the Rubinstein bargaining model with these sorts of preferences is, for instance, much harder than the one we presented in Section 3.7—it may still be useful in isolating the implications of the separability property in certain contexts.

5. Open problems

Time preferences and risky outcomes: The time preference theory presented here treats the preferences over undated outcomes in an ordinal way, and hence it is not suitable for intertemporal choice models in which current and/or future outcomes may obtain in risky environments. An important item in the related future research agenda should thus concern how to extend the present theory to the case where (1) the time projections admit an expected (or non-expected) utility representation, and/or (2) time preferences are defined over lotteries on the entire prize-time space.

Time preferences over consumption streams: We have considered here only the time preferences that are defined on the prize-time space. While this structure is sufficient for some interesting economic applications, it does not apply to numerous dynamic situations, such as the capital accumulation problem, search models, repeated games, etc. An important next step is, therefore, to extend the present analysis to the case of time preferences defined over consumption streams through time. This extension is likely to be highly non-trivial, for it is not even clear what is, if any, the natural generalization of the present model to this case.

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[20] Refs. [3,5,34] prove similar representation results for separable non-transitive preference relations, but while [3] works with preferences on a countable domain, [5] imposes a quite technical conditions on the preferences whose intuitive content is ambiguous. Moreover, as opposed to the algebraic approach adopted by these authors, our approach is topological, and is based on a technique of proof which is entirely different than theirs. It is this technique that allows for deriving the representation from the economically meaningful postulates (A1)–(A4) and (A6).

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Appendix A. Proofs

A.1. An observation on systems of Abel functional equations

As it will be explained in the next subsection, the so-called Abel functional equation plays a crucial role in the proof of Theorem 1. This equation is studied extensively in the literature on iterative functional equations (see [18]). Its general form is

$$\psi(f(x)) - \psi(x) = \text{constant} \quad \text{for all } x \in C,$$

where C is a cone in a Banach space, and the "known" function f is a self-map on C. The "unknown" of the equation is the function $\psi \in \mathbb{R}^X$.

There are relatively fewer studies on systems of such equations (but see [24,37,12]). We state below the main existence result obtained for such systems in the literature, but we need to introduce some terminology first.

For any topological spaces X and Y, we denote the set of all homeomorphisms that map X onto Y by $Hom(X, Y)$, but we write $Hom(X)$ for $Hom(X, X)$. If f is a self-map on X (that is, $f \in X^X$), then $Fix(f)$ denotes the set of all fixed points of f, that is, $Fix(f) := \{x \in X : x = f(x)\}$. We denote the identity function on X by id_X.

Let G be any group, and $S \subseteq G$. The smallest subgroup of G that contains S is called the **group generated** by S, and denoted as $\langle S\rangle$. It easy to show that $s \in \langle S\rangle$ if and only if there are finitely many $s_1, \ldots, s_n \in G$ such that $s = s_1 \cdots s_n$ and either $s_i \in S$ or $s_i^{-1} \in S$ for each $i = 1, \ldots, n$.

Consider the following set of real maps

$$\mathcal{A} := \{f \in Hom(\mathbb{R}) : Fix(f) = \emptyset\},$$

and let $\mathcal{F}$ be any non-empty subset of $\mathcal{A}$. The subgroup generated by $\mathcal{F}$ under the composition operation is denoted by $\langle\mathcal{F}\rangle$. The $\mathcal{F}$-**orbit** of $a \in \mathbb{R}$ is defined as

$$Q_{\mathcal{F}}(a) := \{(f_1 \circ \cdots \circ f_n)(a) : n \in \mathbb{N} \text{ and } f_1, \ldots, f_n \in \mathcal{F}\}.$$

We let $L_{\mathcal{F}}(a)$ denote the set of limit points of $Q_{\mathcal{F}}(a)$, that is, $b \in L_{\mathcal{F}}(a)$ iff there is a sequence (a_m) in $Q_{\mathcal{F}}(a)\backslash\{b\}$ such that $a_m \to b$.

We are now ready to state the following fundamental existence theorem for simultaneous Abel functional equations.

Theorem A.1 (*Jarczyk et al. [12]*). *Let $\mathcal{F}$ be any non-empty subset of $\mathcal{A}$ which contains a map $g \in \mathcal{A}$ with $g > \mathrm{id}_{\mathbb{R}}$. If $\langle\mathcal{F}\rangle$ is Abelian, then*

$$v_g(f) := \sup\left\{\frac{m}{n} : (m, n) \in \mathbb{Z} \times \mathbb{N} \textit{ and } g^m < f^n\right\} \neq 0 \quad \textit{for all } f \in \langle\mathcal{F}\rangle\,.$$

Moreover, if

(i) $\langle \mathcal{F} \rangle$ *is Abelian and the only member of* $\langle \mathcal{F} \rangle$ *that has a fixed point is* $\mathrm{id}_{\mathbb{R}}$, *and*
(ii) *there is some* $a \in \mathbb{R}$ *such that either* $L_{\mathcal{F}}(a) = \{-\infty, \infty\}$ *or* $L_{\mathcal{F}}(a) = \overline{\mathbb{R}}$, *then there exists a continuous bijection* $F : \mathbb{R} \to \mathbb{R}$ *such that*

$$F \circ f - F = v_g(f) \quad \text{for all } f \in \mathcal{F}.$$

Proof. This statement obtains upon combining Propositions 1 and 2, and Theorem 1 of Jarczyk et al. [12]. $\square$

A.2. A primer on the Proof of Theorem 1

This section outlines the basic idea of the proof of Theorem 1, thereby demonstrating the significance of Abel functional equations for the present work. To simplify things, let us focus here on the case where $X := (0, \infty)$. Take any time preference $\succsim$ on $\mathcal{X}$, and for a first pass, assume momentarily that there are only two time periods to consider, 0 and 1. One can show by means of relatively standard arguments that for each $x \in X$ there exists an $f(x) \in X$ such that $(f(x), 0) \sim (x, 1)$. By (A3) we must have $f(x) \leqslant x$ for each x. To flesh out the basic argument, let us assume that $f(x) < x$ for all $x \in X$.

We now pose the following question: Does there exist a constant $\eta(1, 0) > 0$ and a strictly increasing $U : X \to \mathbb{R}_{++}$ such that $U(f(x)) = \eta(1, 0)U(x)$ for each $x > 0$. If the answer is yes to this question, by using (A3) and (A6) we may easily conclude the proof of the desired representation (with, of course, only two time periods). In turn, the answer to this question is yes iff there exist a real number c and a strictly increasing $u : X \to \mathbb{R}$ such that $u(f(x)) - u(x) = c$ for each $x > 0$, that is, $u \circ f - u =$ constant. But the latter equation is none other than an Abel functional equation (where u is the unknown function).

This argument shows how Abel functional equations arise naturally in the present analysis. Of course, considering only two time periods was only for illustrative purposes – in our actual setup this argument would result in a *system* of Abel equations. To see this, for any $s, t \geqslant 0$ define $f_{s,t} : X \to \mathbb{R}_{++}$ through the indifference $(f_{s,t}(x), t) \sim (x, s)$. It is not difficult to show that $f_{s,t}$ is well-defined (even though $\succsim$ is not transitive) by using (A1)–(A3) and continuity. Again, to simplify things, let us assume that $f_{s,t}(x) \neq x$ unless $s = t$. The question now becomes: Given any $s, t \geqslant 0$, can we find a constant $\eta(s, t) > 0$ and a strictly increasing $U : X \to \mathbb{R}_{++}$ (which does not depend on s and t) such that $U(f(x)) = \eta(s, t)U(x)$. The answer to this question is yes iff, for any $s, t \geqslant 0$, there exist a real number $c_{s,t}$ and a strictly increasing $u : X \to \mathbb{R}$ such that $u(f_{s,t}(x)) - u(x) = c_{s,t}$ for each $x > 0$, that is,

$$u \circ f_{s,t} - u = c_{s,t}, \quad s, t \geqslant 0. \tag{A.11}$$

Thus, the main problem in proving Theorem 1 is to "solve" the system of continuum many Abel equations given in (A.11). Of course, this is possible only if there is a certain consistency across these equations. The fact that we are essentially free in the choice of the constants $c_{s,t}$ allows us to tackle this problem by invoking Theorem A. Adopting the terminology of that theorem, we set $\mathcal{F} := \{f_{s,t} : s, t \geqslant 0\}$, $g := f_{0,1}$, and define $c_{s,t} := v_g(f_{s,t})$ for each $s, t \geqslant 0$. (Since we assume in this informal argument that $f_{0,1}(x) \neq x$ for all x, (A3) ensures that $g > \mathrm{id}_{\mathbb{R}}$.) [21] This brings us

[21] In the general case, we would find a specific s and t (by (A1)) such that $s < t$ and $f_{s,t}(x) \neq x$ for at least one $x \in X$. By using (A4), we then find $f_{s,t}(x) \neq x$ for all $x \in X$, that is, by (A3), $f_{s,t} > \mathrm{id}_{\mathbb{R}}$. In this case setting $g := f_{s,t}$ would thus suffice.

well within the coverage of Theorem A, but of course, we still need to verify that $\langle\mathcal{F}\rangle$ is Abelian, and that the only member of $\langle\mathcal{F}\rangle$ with a fixed point is $\mathrm{id}_{\mathbb{R}}$. This is accomplished by using (A4) and (A5). (Condition (ii) of Theorem A is an easy consequence of (A1).)

A.3. Proof of Theorem 1

[$\Rightarrow$] Let $\succsim$ be a time preference on $\mathcal{X}$ that satisfies (A1)–(A6).

Claim 1. *For any $x, y \in X$ and $s, t \geqslant 0$ such that $(x, t) \succ (y, s)$, there exist $z, w \in X$ such that $(x, t) \succ (z, t) \succ (y, s)$ and $(x, t) \succ (w, s) \succ (y, s)$.*

Proof. Define $A := \{\omega \in X : (x, t) \succ (\omega, t)\}$ and $B := \{\omega \in X : (\omega, t) \succ (y, s)\}$. While $A \neq \emptyset$ since $\succsim_t = \succsim_0$ is strictly increasing and X is open, we have $B \neq \emptyset$ because $x \in B$ by hypothesis. Since $\succsim_t$ is upper semicontinuous, A is open in X, and since $\succsim$ is lower semicontinuous, B is open in X. We next claim that $X \subseteq A \cup B$, that is, for any $\omega \in X$, either $(x, t) \succ (\omega, t)$ or $(\omega, t) \succ (y, s)$. This is true, because if $(\omega, t) \succsim (x, t)$, that is, $\omega \succsim_0 x$, is the case, then given that $(x, t) \succ (y, s)$, (A3) implies $(\omega, t) \succ (y, s)$. Thus $X = A \cup B$. Since X is an interval, therefore, we must have $A \cap B \neq \emptyset$. The second claim is proved analogously. $\square$

Claim 2. *For any $y \in X$ and $s, t \geqslant 0$, there exists a unique $x \in X$ such that $(x, t) \sim (y, s)$.*

Proof. By (A2), there exists an $\omega \in X$ such that $(\omega, t) \succsim (y, s)$, so

$$A := \{\omega \in X : (\omega, t) \succsim (y, s)\} \neq \emptyset.$$

Take any increasing $u \in Hom(X, \mathbb{R})$. If $\inf u(A) = -\infty$, then, by (A6), for any $z \in X$ we can find an $\omega \in A$ with $u(z) > u(\omega)$, that is, $(z, t) \succ (\omega, t) \succsim (y, s)$. Thus in this case (A3) implies that $(z, t) \succ (y, s)$ for all $z \in X$, but this violates (A2). It follows that $\inf u(A)$ is a real number, so by surjectivity of u, there exists a $z \in X$ such that $u(x) = \inf u(A)$.

Now if $(y, s) \succ (x, t)$, then there exists a $w \in X$ such that $(y, s) \succ (w, t) \succ (x, t)$ by Claim 1. Since $x \notin A$ and u is injective, we have $\inf u(A) = u(x) \notin u(A)$, so we can find a sequence $(\omega_m) \in A^\infty$ such that $u(\omega_1) > u(\omega_2) > \cdots$ and $u(\omega_m) \to u(x)$. Since, by (A6), u represents $\succsim_t$, we have $u(w) > u(x)$, so there exists an $m_0 \in \mathbb{N}$ such that $u(w) > u(\omega_{m_0}) > u(x)$. Then $(y, s) \succ (w, t) \succ (\omega_{m_0}, t)$, and it follows from (A3) that $\omega_{m_0} \notin A$, a contradiction. Since $\succsim$ is complete, we thus obtain $(x, t) \succsim (y, s)$. However, if $(x, t) \succ (y, s)$, then there exists a $z \in X$ such that $(x, t) \succ (z, t) \succ (y, s)$ by Claim 1, and this contradicts the fact that $u(x) = \inf u(A)$. Therefore, $(x, t) \sim (y, s)$, which settles the existence part of the claim.

To see the uniqueness part, notice that if $(x, t) \sim (y, s)$ and $(x', t) \sim (y, s)$ for some $x, x' \in X$, then (A3) and (A6) entail that neither $x' > x$ nor $x > x'$ can hold. $\square$

Claim 3. *For any $t \geqslant 0$ and $x, y \in X$ with $y \geqslant x$, there exists an $s \geqslant t$ such that $(x, t) \sim (y, s)$.*

Proof. Obviously, we can restrict attention to the case where $y > x$. Let $A := \{r \geqslant 0 : (x, t) \succsim (y, r)\}$, and note that $A \neq \emptyset$ by (A1). Define $s := \inf A$. By upper semicontinuity of $\succsim$, A is closed, and hence $(x, t) \succsim (y, s)$. Moreover, if we had $(x, t) \succ (y, s)$, lower semicontinuity of $\succsim$ would entail the existence of some $r < s$ with $(x, t) \succ (y, r)$, which is impossible by definition of s. It follows that $(x, t) \sim (y, s)$. By (A3) and the fact that $y \succ_0 x$, we must have $s \geqslant t$. $\square$

For any $s, t \geqslant 0$, we define the self-map $\chi_{s,t} : X \to X$ by the statement

$$(y, s) \sim (\chi_{s,t}(y), t), \quad y \in X.$$

By Claim 2, $\chi_{s,t}$ is well-defined for any $s, t \geqslant 0$. The following two claims report further properties of these self-maps.

Claim 4. *$\chi_{s,t}$ is a bijection and $\chi_{s,t}^{-1} = \chi_{t,s}$ for any $s, t \geqslant 0$.*

Proof. Apply Claim 2 and the definitions of $\chi_{s,t}$ and $\chi_{t,s}$. $\square$

Claim 5. *For any increasing $u \in Hom(X, \mathbb{R})$, the map $(s, t, y) \mapsto u(\chi_{s,t}(y))$ is continuous, decreasing in s, increasing in t, and strictly increasing in y.*

Proof. Define $H : \mathbb{R}^2_+ \times X \to \mathbb{R}$ by $H(s, t, y) := u(\chi_{s,t}(y))$. To see that H is decreasing in its first component, take any $y \in X$ and $t \geqslant 0$. If $s_1 > s_2 \geqslant 0$, since $(y, s_1) \sim (\chi_{s_1,t}(y), t)$, then (A3) gives $(\chi_{s_2,t}(y), t) \sim (y, s_2) \succsim (\chi_{s_1,t}(y), t)$. Applying (A3) and (A6), we then find $\chi_{s_2,t}(y) \geqslant \chi_{s_1,t}(y)$, and hence $H(s_2, t, y) = u(\chi_{s_2,t}(y)) \geqslant u(\chi_{s_1,t}(y)) = H(s_1, t, y)$. By Claim 4, this also implies that H is increasing in its second component. Finally, fix any $s, t \geqslant 0$, and take any $z, y \in X$ with $z > y$. Suppose $\chi_{s,t}(y) \geqslant \chi_{s,t}(z)$ is true. Then applying (A3) and (A6) (along with the fact that $(\chi_{s,t}(z), t) \sim (z, s)$) yields $(\chi_{s,t}(y), t) \succsim (z, s)$. But, again by (A3) and (A6) (along with the fact that $(y, s) \sim (\chi_{s,t}(y), t)$), we have $(z, s) \succ (\chi_{s,t}(y), t)$, contradiction. It follows that we must have $\chi_{s,t}(z) > \chi_{s,t}(y)$, and hence $H(s, t, z) > H(s, t, y)$.

To establish that H is continuous, let (s_m) and (t_m) be any non-negative real sequences with finite limits s and t, respectively, and let (y_m) be any real sequence with finite limit y. Since (r_m) must be a bounded sequence, $r_* := \inf r_m$ and $r^* := \sup r_m$ are finite numbers, $r = s, t$. Moreover, $Y := \{y, y_1, y_2, \ldots\}$ is a compact set, so by Weierstrass' Theorem, $y_* := \min u(Y) \in \mathbb{R}$ and $y^* := \max u(Y) \in \mathbb{R}$. By the monotonicity properties of H that we established above, we have $H(s_*, t^*, y^*) \geqslant H(s_m, t_m, y_m) \geqslant H(s^*, t_*, y_*)$ for all m. Then, since X is an interval, $\alpha := \limsup H(s_m, t_m, y_m) \in [H(s_*, t^*, y^*), H(s^*, t_*, y_*)] \subseteq X$. Clearly, there exists a strictly increasing sequence (m_k) of natural numbers such that $H(s_{m_k}, t_{m_k}, y_{m_k}) \to \alpha$. Let $z := u^{-1}(\alpha)$ and $z_k := u^{-1}(H(s_{m_k}, t_{m_k}, y_{m_k})) = \chi_{s_{m_k},t_{m_k}}(y_{m_k})$ for all $k = 1, 2, \ldots$. Since u^{-1} is continuous, then, $z_k \to z$, so $(z_k, t_{m_k}) \to (z, t)$ as $k \to \infty$. We also have $(y_{m_k}, s_{m_k}) \to (y, s)$ and $(y_{m_k}, s_{m_k}) \sim (z_k, t_{m_k})$ as $k \to \infty$. By continuity of $\succsim$ it then follows that $(y, s) \sim (z, t)$, that is, by Claim 2, $z = \chi_{s,t}(y) = u^{-1}(H(s, t, y))$. That is, $\limsup H(s_m, t_m, y_m) = \alpha = H(s, t, y)$. Since one can similarly show that $\liminf H(s_m, t_m, y_m) = H(s, t, y)$, we conclude that H is continuous. $\square$

From now on we will work with a fixed increasing $u \in Hom(X, \mathbb{R})$. Of course, by (A6), this function represents $\succsim_0$. For any $s, t \geqslant 0$, we define

$$f_{s,t} := u \circ \chi_{s,t} \circ u^{-1}.$$

Claim 6. *$f_{s,t} \in Hom(\mathbb{R})$ and $f_{s,t}^{-1} = f_{t,s}$ for any $s, t \geqslant 0$. Moreover, if $t > s \geqslant 0$, then $f_{s,t} \geqslant \mathrm{id}_{\mathbb{R}}$.*

Proof. Fix any $s, t \geqslant 0$. That $f_{s,t}$ is a continuous bijection is immediate from Claims 4 and 5. Moreover, by Claim 4,

$$f_{s,t}^{-1} = u \circ \chi_{s,t}^{-1} \circ u^{-1} = u \circ \chi_{t,s} \circ u^{-1} = f_{t,s},$$

so since $f_{t,s}$ is continuous, $f_{s,t} \in Hom(\mathbb{R})$. Moreover, Claim 4 implies that $\chi_{t,t} = \mathrm{id}_X$ for any $t \geqslant 0$, so by Claim 5, $f_{s,t} \geqslant f_{t,t} = \mathrm{id}_{\mathbb{R}}$. $\square$

Now define

$$\Gamma := \{(s,t) \in \mathbb{R}^2_+ : \chi_{s,t} \neq \mathrm{id}_X\},$$

and note that $(t,t) \notin \Gamma$ for any $t \geqslant 0$. If $\Gamma = \emptyset$, then the proof is completed by letting $U := e^u$ and $\eta := 1$, so we assume in what follows that $\Gamma \neq \emptyset$. Observe that if $(x,s) \sim (x,t)$ for some $x \in X$ and $s,t \geqslant 0$, then (A4) and (A6) ensure that $(z,s) \sim (z,t)$ for all $z \in X$, that is, $(s,t) \notin \Gamma$. (Indeed, we have $(z,s) \sim (\chi_{s,t}(z),t)$, $(x,s) \sim (x,t)$ and $(x,s) \sim (x,s)$, so (A4) entails $(z,s) \sim (\chi_{s,t}(z),s)$, so $z = \chi_{s,t}(z)$ by (A6).) It follows that:

$$\Gamma = \{(s,t) \in \mathbb{R}^2_+ : Fix(\chi_{s,t}) = \emptyset\}, \tag{A.12}$$

and hence

$$\mathcal{F} := \{f_{s,t} : (s,t) \in \Gamma\} \subseteq \{f \in Hom(\mathbb{R}) : Fix(f) = \emptyset\}$$

by Claim 6. We shall demonstrate below that $\mathcal{F}$ satisfies the other requirements of Theorem A as well. We need the following auxiliary observation for that purpose.

Claim 7. *For any $x, y, z, w \in X$ and $s_1, s_2, t_1, t_2 \in T$, if*

$$\begin{array}{l} (x,t_1) \sim (y,s_1) \\ \qquad\qquad \textit{and } (w,t_2) \sim (y,s_2) \quad \textit{then } (z,t_2) \sim (x,s_2). \\ (z,t_1) \sim (w,s_1) \end{array}$$

Proof. Without loss of generality, let $w > y$, and use Claim 3 to find an $s \geqslant 0$ such that $(y,s_1) \sim (w,s)$. Applying (A5), then, we find $(x,s_1) \sim (z,s)$. Combining these observations with the hypothesis that $(y,s_2) \sim (w,t_2)$ and (A4), then, we obtain $(x,s_2) \sim (z,t_2)$. $\square$

Fix an arbitrary $(s^*,t^*) \in \Gamma$ with $s^* < t^*$, and define

$$g := f_{s^*,t^*}.$$

By Claim 6, $g \geqslant \mathrm{id}_{\mathbb{R}}$, so since $(s^*,t^*) \in \Gamma$, we must have $g > \mathrm{id}_{\mathbb{R}}$.

Now take any $(s_1,t_1), (s_2,t_2) \in \Gamma$, and any $y \in X$. Let $w := \chi_{s_2,t_2}(y)$, $z := \chi_{s_1,t_1}(w)$ and $x := \chi_{s_1,t_1}(y)$. Then by definition of χ_{s_2,t_2} and Claim 7, we have

$$\chi_{s_1,t_1}(\chi_{s_2,t_2}(y)) = \chi_{s_1,t_1}(w) = z = \chi_{s_2,t_2}(x) = \chi_{s_2,t_2}(\chi_{s_1,t_1}(y)).$$

Since y is arbitrary in X, this shows that χ_{s_1,t_1} and χ_{s_2,t_2} commute, and it follows easily from this observation that so do f_{s_1,t_1} and f_{s_2,t_2}. Since (s_1,t_1) and (s_2,t_2) are arbitrary in Γ, it follows that $\langle\mathcal{F}\rangle$ must be an Abelian group.

We will next show that if $f \in \langle\mathcal{F}\rangle$ and $Fix(f) \neq \emptyset$, then $f = \mathrm{id}_{\mathbb{R}}$. To this end, we need to introduce some further terminology. Let $n \in \{2,3,\ldots\}$, and denote the generic element of $\mathbb{R}^{2n}_+$ by $(s_i,t_i)_{i=1}^n$. For any $x \in X$, we say that $(s_i,t_i)_{i=1}^n$ is an x-**cycle** if there exist $y_1,\ldots,y_{n-1} \in X$ such that

$$(x,t_1) \sim (y_1,s_1), (y_1,t_2) \sim (y_2,s_2), \ldots, (y_{n-1},t_n) \sim (x,s_n).$$

Claim 8. *For any $n \in \{2, 3, \ldots\}$ and $x \in X$, if $(s_i, t_i)_{i=1}^n$ is an x-cycle, then $(s_i, t_i)_{i=1}^n$ is a z-cycle for any $z \in X$.*

Proof. Take any $z \in X$, $n \geqslant 2$, and an x-cycle $(s_i, t_i)_{i=1}^n$. By applying Claim 2 n times we find $w_1, \ldots, w_n \in X$ such that

$$(z, t_1) \sim (w_1, s_1), (w_1, t_2) \sim (w_2, s_2), \ldots, (w_{n-1}, t_n) \sim (w_n, s_n).$$

By Claim 3, there exist $s, t \geqslant 0$, $(x, t) \sim (z, s)$. Then Claim 7 implies $(y_1, t) \sim (w_1, s)$. Applying Claim 7 one more time, we then get $(y_2, t) \sim (w_2, s)$. Proceeding inductively, therefore, we obtain $(x, t) \sim (w_n, s)$. Since $(x, t) \sim (z, s)$, Claim 2 then gives $z = w_n$, so $(s_i, t_i)_{i=1}^n$ is a z-cycle. □

Now take any $f \in \langle \mathcal{F} \rangle$. By definition, there exist an $n \in \mathbb{N}$ and $(s_1, t_1), \ldots, (s_n, t_n) \in \Gamma$ such that $f = f_{s_1,t_1} \circ \cdots \circ f_{s_n,t_n}$. If $n = 1$, then $Fix(f) = \emptyset$ by definition of $\mathcal{F}$. If $n \geqslant 2$ and $a = f(a)$ for some real a, then

$$a = \left(u \circ \chi_{s_1,t_1} \circ \cdots \circ \chi_{s_n,t_n} \circ u^{-1}\right)(a),$$

so letting $x := u^{-1}(a) \in X$, we then get $x = \left(\chi_{s_1,t_1} \circ \cdots \circ \chi_{s_n,t_n}\right)(x)$. But by Claim 8, this implies that $\chi_{s_1,t_1} \circ \cdots \circ \chi_{s_n,t_n} = \mathrm{id}_X$, so it follows that $f = u \circ \mathrm{id}_X \circ u^{-1} = \mathrm{id}_{\mathbb{R}}$. That is, the only member of $\langle \mathcal{F} \rangle$ that has a fixed point is $\mathrm{Id}_{\mathbb{R}}$.

Finally, take an arbitrary real number, say 0, and let $x := u^{-1}(0)$. For any $b \in \mathbb{R}$, let $y := u^{-1}(b)$, and observe that, by (A1), there exist $(s, t) \in \Gamma$ such that $\chi_{s,t}(y) = x$. But then $f_{s,t}(0) = u(\chi_{s,t}(u^{-1}(0))) = b$. Since b is an arbitrary real number here, it follows that $\bigcup\{f(0) : f \in \mathcal{F}\} = \mathbb{R}\backslash\{0\}$, and hence $Q_{\mathcal{F}}(a) := \bigcup\{f(0) : f \in \langle \mathcal{F} \rangle\} = \mathbb{R}$.

We have verified that $\mathcal{F}$ satisfies all requirements of Theorem A. Define $\varphi : \Gamma \to \mathbb{R}$ by $\varphi(s, t) := v_g(f_{s,t})$, where v_g is defined as in Theorem A. By this theorem, $\varphi(s, t) \neq 0$ for all $(s, t) \in \Gamma$, and there exists a continuous bijection $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(f_{s,t}(a)) - F(a) = \varphi(s, t) \quad \text{for all } (a, (s, t)) \in \mathbb{R} \times \Gamma. \tag{A.13}$$

Claim 9. *For any $(s, t) \in \Gamma$, we have*

$$\varphi(s, t) = -\varphi(t, s) \quad \textit{and} \quad (s - t)\varphi(s, t) < 0.$$

Proof. Let $(s, t) \in \Gamma$, and take any $a \in \mathbb{R}$. By (A.13) and Claim 6, we have

$$\begin{aligned} F(f_{s,t}(a)) - F(a) &= \varphi(s, t) \\ &= F(f_{s,t}(f_{s,t}^{-1}(a))) - F(f_{s,t}^{-1}(a)) \\ &= F(a) - F(f_{t,s}(a)) \\ &= -\varphi(t, s). \end{aligned}$$

To prove the second claim, assume that $s > t$. Then we have $g > \mathrm{id}_{\mathbb{R}} > f_{s,t}$, so for any $(m, n) \in \mathbb{Z}_+ \times \mathbb{N}$ we have $g^m > \mathrm{id}_{\mathbb{R}} > f_{s,t}^n$. Thus by the first part of Theorem A, we have $\varphi(s, t) = v_g(f_{s,t}) < 0$, as we sought. □

Take any $a \in \mathbb{R}$ and let $b := g(a)$. By Claim 9, we have $F(b) = F(a) + \varphi(s^*, t^*) > F(a)$. Since $g > \mathrm{id}_{\mathbb{R}}$, we have $b > a$, so it follows that F cannot be strictly decreasing. Since F is bijective, therefore, it must be strictly increasing. We now define $V : X \to \mathbb{R}$ by $V := F \circ u$, and extend φ to $\mathbb{R}^2_+$ by setting $\varphi|_{\mathbb{R}^2_+ \setminus \Gamma} := 0$. It will be shown next that V and φ satisfy the requirements made of them in Theorem 1.

Because F is continuous, surjective and strictly increasing, V is a continuous bijection that represents $\succsim_0$. Since any such function on a real interval has a continuous inverse, we also have $V \in Hom(X, \mathbb{R})$. Moreover, since $f_{t,t} = \mathrm{id}_{\mathbb{R}}$ for all $t \geqslant 0$ by Claim 4, we have, for any fixed $a \in \mathbb{R}$, $\varphi(s, t) = F(f_{s,t}(a)) - F(a)$ for all $s, t \geqslant 0$. Since F is continuous and strictly increasing, Claim 5 and the definition of $f_{s,t}$ immediately show that φ is a continuous function which is decreasing in its first component and is continuous. Moreover, by Claim 9, we have $\varphi(s, t) = -\varphi(t, s)$ for all $s, t \geqslant 0$.

Now define $U := e^V$ and $\eta := e^{\varphi}$. Obviously, $U \in Hom(X, \mathbb{R}_{++})$ and $\eta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ is a continuous function that satisfies (i) and (ii) of Theorem 1. To verify (4), take any $(x, t), (y, s) \in X \times \mathbb{R}_+$. Clearly, $(x, t) \succ (\sim)(y, s)$ holds if and only if $(x, t) \succ (\sim)(y, s) \sim (\chi_{s,t}(y), t)$. Thus, by completeness of $\succsim$, (A3), we have

$$(x,t) \left\{ \begin{matrix} \succ \\ \sim \end{matrix} \right\} (y,s) \quad \text{iff } (x,t) \left\{ \begin{matrix} \succ \\ \sim \end{matrix} \right\} (\chi_{s,t}(y), t).$$

Since U represents $\succsim_0 = \succsim_t$, we thus have

$$(x,t) \left\{ \begin{matrix} \succ \\ \sim \end{matrix} \right\} (y,s) \quad \text{iff } \frac{U(x)}{U(y)} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} \frac{U(\chi_{s,t}(y))}{U(y)}.$$

But if $a := V(y)$, then by (A.13),

$$\begin{aligned} \frac{U(\chi_{s,t}(y))}{U(y)} &= \exp\big(V(\chi_{s,t}(y)) - V(y)\big) \\ &= \exp\Big(F(V(\chi_{s,t}(V^{-1}(a)))) - F(V(y))\Big) \\ &= \exp\big(F(f_{s,t}(a)) - F(a)\big) \\ &= \exp \varphi(s,t) \\ &= \eta(s,t) \end{aligned}$$

provided that $(s, t) \in \Gamma$. If, on the other hand, $(s, t) \notin \Gamma$, then $\chi_{s,t} = \mathrm{id}_X$, so

$$\frac{U(\chi_{s,t}(y))}{U(y)} = \exp\big(V(\chi_{s,t}(y)) - V(y)\big) = 1 = \exp \varphi(s,t) = \eta(s,t).$$

Combining these observations, we conclude that $\frac{U(\chi_{s,t}(y))}{U(y)} = \eta(s, t)$ for al $s, t \geqslant 0$, and hence

$$(x,t) \left\{ \begin{matrix} \succ \\ \sim \end{matrix} \right\} (y,s) \quad \text{iff } U(x) \left\{ \begin{matrix} > \\ = \end{matrix} \right\} \eta(s,t) U(y),$$

which establishes (4).

It remains to prove that $\eta(\infty, t) = 0$ for any $t \geqslant 0$. To this end, fix any $t \geqslant 0$ and choose any $\varepsilon > 0$. Now let $x := U^{-1}(\varepsilon)$ and $y := U^{-1}(1)$, and apply (A1) to find an $s \geqslant 0$ such that $(x, t) \succsim (y, s)$, that is, $\varepsilon = U(x) \geqslant \eta(s, t)U(y) = \eta(s, t)$. Since $\varepsilon > 0$ is arbitrary here, and $\eta(\cdot, t)$ is decreasing, we must have $\lim_{s\to\infty} \eta(s, t) = 0$, as we sought.

[$\Leftarrow$] Assume that $\succsim$ is a binary relation on $\mathcal{X}$ such that there exist an increasing $U \in Hom(X, \mathbb{R}_{++})$ and a continuous map $\eta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ such that (4) holds for all (x, t) and (y, s) in $\mathcal{X}$, and η

satisfies properties (i)–(ii) asserted in Theorem 1. It is obvious that $\succsim$ is complete and continuous. On the other hand, by property (ii), we have $\eta(t,t) = 1$, so by (4), U represents $\succsim_t$ for all $t \geqslant 0$. Thus, $\succsim$ is a time preference on $\mathcal{X}$. Given (4), $\succsim$ satisfies (A4) and (A5) obviously, (A1) because η is surjective, (A2) because U is surjective, (A3) and (A6) because U is strictly increasing and η satisfies (i) and (ii).

A.4. Proof of Proposition 1

"If" part is trivial, so we focus only on the "only if" part of the claim. Let $\succsim$ be a time preference on $\mathcal{X}$, which is represented by both (U, η) and (V, ζ). Clearly, both U and V are homeomorphisms on X that represent $\succsim_0$, so there exists a strictly increasing $f \in Hom(\mathbb{R}_{++})$ such that $V = f \circ U$. For any $y \in X$ and $s, t \geqslant 0$, by surjectivity of U, there exists an $x \in X$ such that $U(x) = \eta(s,t)U(y)$. Since $(x,t) \sim (y,s)$, this implies also that $f(U(x)) = \zeta(s,t)f(U(y))$. Therefore,

$$f(\eta(s,t)U(y)) = \zeta(s,t)f(U(y)) \quad \text{for all } y \in X \text{ and } s,t \geqslant 0. \tag{A.14}$$

Choosing $y := U^{-1}(1)$ and letting $b := f(1) > 0$, we then have

$$f(\eta(s,t)) = b\zeta(s,t) \quad \text{for all } s,t \geqslant 0. \tag{A.15}$$

Letting $g := f/b$ and using this observation in (A.14),

$$f(\eta(s,t)U(y)) = g(\eta(s,t))f(U(y)) \quad \text{for all } y \in X \text{ and } s,t \geqslant 0.$$

Conditions (i) and (ii) guarantee that η is surjective. Thus, dividing both sides of the above equation by b, and using the surjectivity of η and U, we find that $g(mn) = g(m)g(n)$ for all $m, n > 0$, that is, g satisfies the multiplicative Cauchy functional equation on $\mathbb{R}_{++}$. Then, given that g is strictly increasing, there must exist an $a > 0$ such that $g(\alpha) = \alpha^a$ for all $\alpha > 0$. Therefore, since $V = f \circ U$, we have $V = bU^a$. Moreover, it follows from (15) that $\zeta = \eta^a$.

A.5. Proofs of Corollaries 1–5

In any one of the Corollaries 1–5, the non-trivial part of the assertion concerns the "only if" part, so we focus on this part alone.

Proof of Corollary 1. Apply Theorem 1 to find a (U, η) that represents $\succsim$ (with U being increasing). Define $D : \mathbb{R}_+ \to (0, 1]$ by $D(k) := \eta(k, 0)$. Then, by continuity of η and conditions (i) and (ii) of Theorem 1, D is decreasing, continuous, $D(0) = 1$ and $D(\infty) = 0$. Finally, by (5) and (ii) of Theorem 1, we have $\eta(s,t) = \eta(s,0)\eta(0,t) = D(s)/D(t)$ for all $s,t \geqslant 0$, so (6) follows from (4) for all $(x,t), (y,s) \in \mathcal{X}$. $\square$

Poof of Corollary 2. Apply Theorem 1 to find a (U, η) that represents $\succsim$ (with U being increasing). Define $\zeta : \mathbb{R} \to \mathbb{R}_{++}$ as $\zeta(k) := \eta(k, 0)$ if $k \geqslant 0$, and $\zeta(k) := \eta(0, -k)$ if $k < 0$. Then ζ is continuous, decreasing, $\zeta(\infty) = 0$, and $\zeta(t) = 1/\zeta(-t)$ for all $t \geqslant 0$, because η is continuous and satisfies conditions (i)–(ii) of Theorem 1. Finally, (7) follows from (4) and stationarity of $\succsim$. $\square$

Proof of Corollary 3. Let U and D be as found by Corollary 1. By stationarity of $\succsim$, we have $D(t)/D(s) = D(t-s)/D(0)$ for all $t \geqslant s \geqslant 0$. Changing variables and recalling that $D(0) = 1$, we find $D(r+s) = D(r)D(s)$ for all $r, s \geqslant 0$. Since D is continuous, by a well-known theorem of Cauchy, there must then exist a $\delta \in \mathbb{R}$ such that $D(t) = \delta^t$ for all $t \geqslant 0$. Since $D \geqslant 0$ and $D(\infty) = 0$, we have $0 < \delta < 1$. □

Proof of Corollary 4. Straightforward. □

Proof of Corollary 5. Apply Theorem 1 on the outcome space (x_*, x^*) to find an increasing homeomorphism $U : (x_*, x^*) \to \mathbb{R}_{++}$ and a continuous map $\eta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ such that (4) and (i) and (ii) of Theorem 1 hold for all $x, y \in X \backslash \{x_*\}$ and $s, t \geqslant 0$. We next extend U to X by setting $U(x_*) := 0$. All we need to show is that (4) holds when $x_* \in \{x, y\}$. In turn, to establish this, it is enough to show that x_* satisfies (9).

Fix any $s, t \in [0, \infty)$. To derive a contradiction, assume first that $(x_*, s) \succ (x_*, t)$. Since, by (A2), there exists a $y \in X \backslash \{x_*\}$ such that $(x_*, t) \succsim (y, s)$, (A3) yields in this case that $x_* \succsim_s y$, which contradicts (A6) since $\succsim_s = \succsim_0$ and $y > x_*$. Therefore, we have $(x_*, t) \sim (x_*, s)$ as is sought. The proof of the second claim in (9) is similar. Indeed, if $(x_*, s) \succsim (x, t)$, then (A3), (A6) and the fact that $x > x_*$ imply $(x_*, s) \succ (x_*, t)$, which contradicts the first part of (9). □

A.6. *Proof of Theorem 2*

The proof of the "if" part of this result is identical to that of the "if" part of Theorem 1, while the proof of the uniqueness claim is identical to that of Proposition 1. We will thus only prove here the "only if" part. To this end, let us fix the numbers $-\infty \leqslant a_i < b_i \leqslant \infty$, $i = 1, \ldots, n$, and let $X := (a_1, b_1) \times \cdots \times (a_n, b_n)$. In what follows $\succsim$ stands for a time preference on $\mathcal{X}$ that satisfies (A1)–(A5).

Claim 10. *There exists a strictly increasing and continuous surjection $w : X \to \mathbb{R}_{++}$ that represents $\succsim_0$. Moreover, w^{-1} admits a continuous selection f (that is, a continuous map $f : \mathbb{R}_{++} \to X$ with $f(\omega) \in w^{-1}(\omega)$ for all $\omega > 0$).*

Proof. By using the Debreu Representation Theorem one can show that there exists a continuous surjection $w : X \to \mathbb{R}_{++}$ that represents $\succsim_0$.[22] For each $i = 1, \ldots, n$, take any strictly increasing sequence $(a_{i,m})$ in $(a_i, (a_i + b_i)/2)$ such that $a_{i,m} \to a_i$. Similarly, for each i, take any strictly decreasing sequence $(b_{i,m})$ in $((a_i + b_i)/2, b_i)$ such that $b_{i,m} \to b_i$. Let

$$\mathbf{a}_m := (a_{1,m}, \ldots, a_{n,m}) \quad \text{and} \quad \mathbf{b}_m := (b_{1,m}, \ldots, b_{n,m}), \quad m = 1, 2, \ldots .$$

We define $f_1 : [w(\mathbf{a}_1), w(\mathbf{b}_1)] \to X$ by the equation

$$\{f_1(\omega)\} = \mathrm{co}\{\mathbf{a}_1, \mathbf{b}_1\} \cap w^{-1}(\omega).$$

Since w is a strictly increasing and continuous surjection, a straightforward argument based on the Intermediate Value Theorem shows that f is well-defined and continuous. Now define

[22] By the Debreu Representation Theorem, there exists a continuous map $v : X \to \mathbb{R}$ that represents $\succsim_0$. Since X is open, v must be strictly increasing, and neither $\alpha := \inf v(X)$ nor $\beta := \sup v(X)$ belongs to $v(X)$. We define $w : X \to \mathbb{R}_{++}$ by $w(x) := \frac{v(x)-\alpha}{\beta - v(x)}$ if $\alpha, \beta \in \mathbb{R}$, by $w(x) := \frac{1}{\beta - v(x)}$ if $\alpha = -\infty$ and $\beta \in \mathbb{R}$, by $w(x) := v(x) - \alpha$ if $\alpha \in \mathbb{R}$ and $\beta = \infty$, and by $w(x) = e^{v(x)}$ otherwise.

$f_2 : [w(\mathbf{a}_2), w(\mathbf{b}_2)] \to X$ by the equation

$$\{f_2(\omega)\} := \begin{cases} \mathrm{co}\{\mathbf{a}_2, \mathbf{a}_1\} \cap w^{-1}(\omega) & \text{if } w(\mathbf{a}_2) \leqslant \omega < w(\mathbf{a}_1), \\ \{f_1(\omega)\} & \text{if } w(\mathbf{a}_1) \leqslant \omega \leqslant w(\mathbf{b}_1), \\ \mathrm{co}\{\mathbf{b}_1, \mathbf{b}_2\} \cap w^{-1}(\omega) & \text{if } w(\mathbf{b}_1) < \omega \leqslant w(\mathbf{b}_2). \end{cases}$$

It is again easily checked that f_2 is well-defined and continuous. Proceeding this way inductively, we obtain a sequence (f_m) such that, for each $m = 1, 2, \ldots,$

(i) $f_m : [w(\mathbf{a}_m), w(\mathbf{b}_m)] \to X$ is a continuous function;
(ii) f_{m+1} is an extension of f_m;
(iii) $f_m(\omega) \in w^{-1}(\omega)$ for all $w(\mathbf{a}_m) \leqslant \omega \leqslant w(\mathbf{b}_m)$.

To complete the proof, define $f : \mathbb{R}_{++} \to X$ by

$$f(\omega) := f_m(\omega) \quad \text{for any } m \in \mathbb{N} \text{ such that } w(\mathbf{a}_m) \leqslant \omega \leqslant w(\mathbf{b}_m).$$

Since w is strictly increasing, the choice of $(\mathbf{a}_m)$ and $(\mathbf{b}_m)$ warrants that $\lim w(\mathbf{a}_m) = \inf w(X) = 0$, and $\lim w(\mathbf{b}_m) = \sup w(X) = \infty$. Thus, by continuity of w, we have $(\lim w(\mathbf{a}_m), \lim w(\mathbf{b}_m)) = \mathbb{R}_{++}$. Consequently, (ii) guarantees that f is well-defined. Moreover, by (i), f is a continuous function. Finally, (iii) ensures that f is a selection from the correspondence w^{-1}. □

Let w and f be as found in Claim 10, and define the binary relation $\trianglerighteq$ on $\mathbb{R}_{++} \times [0, \infty)$ as

$$(a, t) \trianglerighteq (b, s) \quad \text{iff } (f(a), t) \succsim (f(b), s).$$

It is obvious that $\trianglerighteq$ is complete, $\trianglerighteq_0$ is complete and transitive, and $\trianglerighteq_0 = \trianglerighteq_t$ for each t. Moreover, continuity of f implies that of $\trianglerighteq$, and hence $\trianglerighteq$ is a time preference on $\mathbb{R}_{++} \times [0, \infty)$. One can also check easily that $\trianglerighteq$ satisfies (A1)–(A6). By Theorem 1, therefore, there exists an increasing $V \in \mathit{Hom}(\mathbb{R}_{++})$ and a continuous map $\eta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ such that

$$(a, t) \trianglerighteq (b, s) \quad \text{iff } V(a) \geqslant \eta(s, t) V(b)$$

for all $a, b > 0$ and $s, t \geqslant 0$, and conditions (i) and (ii) of Theorem 2 hold. Since, for any $x, y \in X$, we have $f(w(x)) \sim_0 x$ and $f(w(y)) \sim_0 y$, an easy application of (A3) yields

$$(x, t) \succsim (y, s) \quad \text{iff } (f(w(x)), t) \trianglerighteq (f(w(y)), s)$$

for all $s, t \geqslant 0$. But

$$(f(w(x)), t) \trianglerighteq (f(w(y)), s) \quad \text{iff } (w(x), t) \trianglerighteq (w(y), s) \ \text{ iff } V(w(x)) \geqslant \eta(s, t) V(w(y)),$$

so, by letting $U := V \circ w$, we find $(x, t) \succsim (y, s)$ iff $U(x) \geqslant \eta(s, t) U(y)$ for all $x, y \in X$ and $s, t \geqslant 0$. Since U is a strictly increasing and continuous surjection from X onto $\mathbb{R}_{++}$, we are done.

A.7. Proof of Theorem 3

The proof of the "if" part of this result is easy, so we focus only on the "only if" part. Let $\succsim$ be a time preference on $\mathcal{X}$ that satisfies (A1)–(A4) and (A6), and adopt the notation of the proof of Theorem 1. We shall need the following auxiliary fact in the subsequent arguments.

Claim 11. *$f_{s,t}$ is a strictly increasing surjection for any $s, t \geqslant 0$. Moreover, if $f_{s,t}(a) = f_{s',t'}(a)$ for some $a \in \mathbb{R}$ and $s, s', t, t' \geqslant 0$, then $f_{s,t} = f_{s',t'}$.*

Proof. For any $s, t \geqslant 0$, Claim 5 entails that $\chi_{s,t}$ is strictly increasing, and hence $f_{s,t}$ must be a strictly increasing surjection (Claim 6). To prove the second assertion assume that $f_{s,t}(a) = b = f_{s',t'}(a)$ for some $a, b \in \mathbb{R}$ and $s, s', t, t' \geqslant 0$. Let $y := u^{-1}(a)$ and $x := u^{-1}(b)$. Then, $u(\chi_{s,t}(y)) = u(x) = u(\chi_{s',t'}(y))$, that is, $(y, s) \sim (x, t)$ and $(y, s') \sim (x, t')$. Since, for any $z \in X$ we have $(z, s) \sim (\chi_{s,t}(z), t)$, therefore, (A4) entails $(z, s') \sim (\chi_{s,t}(z), t')$, that is, $\chi_{s,t}(z) = \chi_{s',t'}(z)$ by (A3). Hence, $u(\chi_{s,t}(z)) = u(\chi_{s',t'}(z))$ for all $z \in X$, and the claim follows from the fact that u is an injection. □

Define $\eta : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ as

$$\eta(s, t) := e^a \quad \text{for any } a \in \mathbb{R} \text{ with } f_{s,t}(-a) = a.$$

The first part of Claim 11 ensures that, for any $s, t \geqslant 0$, the graph of $f_{s,t}$ intersects the inverse diagonal $\{(-a, a) : a \in \mathbb{R}\}$ exactly once. Thus η is well-defined. Moreover, for any $s, t \geqslant 0$, if $\eta(s, t) = e^a$, then $-a = f^{-1}_{s,t}(a) = f_{t,s}(a)$, so $\eta(t, s) = e^{-a} = 1/\eta(s, t)$. Now fix any $t \geqslant 0$. (A3) ensures that $\eta(\cdot, t)$ is decreasing. To show that $\eta(\infty, t) = 0$, take any $a < 0$, and define $x := u^{-1}(a)$ and $y := u^{-1}(-a)$. By Claim 3, there exists an $s_a \geqslant 0$ such that $(x, t) \sim (y, s_a)$. Thus $\chi_{s_a,t}(y) = x$, and hence, $f(-a) = u(\chi_{s_a,t}(y)) = u(x) = a$, that is, $\eta(s_a, t) = e^a$. Since a is an arbitrary negative number here, it follows that $\eta(\infty, t) = 0$. (This also shows that η is surjective.)

Next, define $F : \mathbb{R}^2 \to \mathbb{R}_{++}$ as

$$F(\alpha, \beta) := \eta(s, t) \quad \text{for any } s, t \geqslant 0 \text{ with } f_{t,s}(\alpha) = \beta.$$

The second part of Claim 11 readily establishes that F is well-defined. Moreover, surjectivity of η readily ensures that $F(\cdot, \beta)$ is surjective for any $\beta \in \mathbb{R}$. Finally, for any $\alpha, \beta \in \mathbb{R}$, $F(\alpha, \beta) = \eta(s, t)$ implies $f_{t,s}(\alpha) = \beta$ so that $f_{s,t}(\beta) = \alpha$ (because $f^{-1}_{t,s} = f_{s,t}$), and we thus have $F(\beta, \alpha) = \eta(t, s) = 1/\eta(s, t) = 1/F(\alpha, \beta)$.

Now fix any (x, t) and (y, s) in $\mathcal{X}$. If $(x, t) \sim (y, s)$, then $\chi_{s,t}(x) = y$ and hence $f_{t,s}(u(x)) = u(y)$, that is, $F(u(x), u(y)) = \eta(s, t)$. Now suppose $(x, t) \succ (y, s)$, and let $F(u(x), u(y)) = \eta(s', t')$ which means $y = \chi_{t',s'}(x)$ by definition of F. Let $z := \chi_{s,t}(y)$ so that $y = \chi_{t,s}(z)$ and $F(u(z), u(y)) = \eta(s, t)$. Since $(x, t) \succ (y, s) \sim (z, t)$, (A3) implies that $x > z$ and since $\chi_{t,s}$ is strictly increasing, we have $\chi_{t',s'}(x) = y = \chi_{t,s}(z) < \chi_{t,s}(x)$. It follows from (the proof of) the second part of Claim 11, therefore, $f_{t',s'} < f_{t,s}$, that is, $f_{s',t'} > f_{s,t}$. We now claim that $\eta(s', t') > \eta(s, t)$. If this was not the case, then there would exist real numbers $a \geqslant b$ such that $a = f_{s,t}(-a)$ and $b = f_{s',t'}(-b)$, and since $f_{s',t'}$ is increasing and $f_{s',t'} > f_{s,t}$, we would obtain

$$b = f_{s',t'}(-b) \geqslant f_{s',t'}(-a) > f_{s,t}(-a) = a,$$

a contradiction. Thus, $F(u(x), u(y)) = \eta(s', t') > \eta(s, t)$, as we sought. Conversely, if $F(u(x), u(y)) \geqslant \eta(s, t)$ but $(y, s) \succ (x, t)$, then, by what is just established, we get

$$\frac{1}{F(u(x), u(y))} = F(u(y), u(x)) > \eta(t, s) = \frac{1}{\eta(s, t)},$$

contradicting $F(u(x), u(y)) \geqslant \eta(s, t)$. Thus, we conclude that (10) holds for all $x, y \in X$ and $s, t \geqslant 0$. In turn, it follows from this representation and (A3) that $F(\cdot, \beta)$ is strictly increasing and $\eta(\cdot, t)$ is decreasing for all $\beta \in \mathbb{R}$ and $t \geqslant 0$.

It remains to show that both η and F are continuous functions. We shall do this only for η as the argument for F is analogous. Take any convergent sequences (s_m) and (t_m) in $\mathbb{R}_+$, and let $s := \lim s_m$ and $t := \lim t_m$. By definition of η, there exists a real sequence (a_m) such that $f_{s_m,t_m}(-a_m) = a_m$ and $\eta(s_m, t_m) = e^{a_m}$ for each m. Define $x_m := u^{-1}(-a_m)$ and $y_m := u^{-1}(a_m)$ so that $(x_m, t_m) \sim (y_m, s_m)$ for each m. Now let $a := \limsup a_m$, and assume, to get a contradiction, $a = \infty$. In this case there exists a subsequence (a_{m_k}) of (a_m) that diverges to ∞. Notice that

$$(x_{m_k}, t_*) \succsim (x_{m_k}, t_{m_k}) \sim (y_{m_k}, s_{m_k}) \succsim (y_{m_k}, s^*), \quad k = 1, 2, \ldots,$$

where $t_* := \inf\{t_m : m = 1, 2, \ldots\}$ and $s^* := \sup\{s_m : m = 1, 2, \ldots\}$. Now take any $x \in X$, and use Claim 2 to find a $y \in X$ such that $(x, t_*) \sim (y, s^*)$. Then, if k is large enough to guarantee that $-a_{m_k} < u(x)$ and $a_{m_k} > u(y)$, we have $(x_{m_k}, t_*) \prec (y_{m_k}, s^*)$ by (A3), contradiction. A similar argument establishes that $a = -\infty$ is impossible, so a is a real number. But then, by Claim 5 and continuity of u,

$$a = \lim a_{m_k} = \lim f_{s_{m_k},t_{m_k}}(-a_{m_k}) = \lim u(\chi_{s_{m_k},t_{m_k}}(x_{m_k})) = u(\chi_{s,t}(u^{-1}(a))) = f_{s,t}(a).$$

Thus $\limsup \eta(s_m, t_m) = e^a = \eta(s, t)$. Since one can similarly show that $\liminf \eta(s_m, t_m) = \eta(s, t)$, we conclude that η is continuous.

References

[1] M. Abdellaoui, A genuine rank-dependent generalization of the von Neumann–Morgenstern expected utility theorem, Econometrica 70 (2002) 717–736.
[2] J. Benhabib, A. Bisin, A. Schotter, Discounting and time preferences: an experimental analysis, Mimeo, New York University, 2002.
[3] D. Bouyssou, Some remarks on the notion of compensation in MCDM, Europ. J. Operational Res. 26 (1986) 150–160.
[4] G. Debreu, Topological methods in cardinal utility theory, in: K. Arrow, S. Karlin, P. Suppes (Eds.), Mathematical Methods in the Social Sciences, Stanford University Press, Stanford, 1960.
[5] P. Fishburn, Nontransitive additive conjoint measurement, J. Math. Psych. 35 (1991) 1–40.
[6] P. Fishburn, Nontransitive preferences in decision theory, J. Risk Uncertainty 4 (1991) 113–134.
[8] P. Fishburn, A. Rubinstein, Time preference, Int. Econ. Rev. 23 (1982) 677–694.
[9] S. Frederick, G. Loewenstein, T. O'Donoghue, Time discounting: a critical review, J. Econ. Lit. 40 (2002) 351–401.
[10] A. Geir, Individual and collective time consistency, Rev. Econ. Stud. 54 (1997) 427–443.
[11] S. Goldman, Consistent plans, Rev. Econ. Stud. 47 (1980) 533–537.
[12] W. Jarczyk, K. Loskot, M. Zdun, Commuting functions and simultaneous Abel equations, Ann. Polonici Math. 62 (1994) 119–135.
[13] R. Keeney, H. Raiffa, Decisions with Multiple Objectives, Wiley, New York, 1976.
[14] T.C. Kocherlakota, Reconsideration-proofness: a refinement for infinite horizon time inconsistency, Games and Econ. Behav. 15 (1996) 33–54.
[15] T.C. Koopmans, Stationary ordinal utility and impatience, Econometrica 28 (1960) 287–309.
[16] T.C. Koopmans, P. Diamond, R. Williamson, Stationary utility and time perspective, Econometrica 32 (1964) 82–100.
[17] D. Krantz, D. Luce, P. Suppes, A. Tversky, Foundations of Measurement, vol. 1, Academic Press, New York, 1971.
[18] M. Kuczma, B. Choczewski, R. Ger, Iterative Functional Equations, Cambridge University Press, Cambridge, 1990.
[19] D. Laibson, Golden eggs and hyperbolic discounting, Quart. J. Econ. 112 (1997) 443–477.
[23] P. Manzini, M. Mariotti, A vague theory of choice over time, Adv. Theoret. Econ. 6(1), Article 6.2006.
[24] F. Neuman, Simultaneous solution of a system of Abel equations and differential equations with several deviations, Czechoslovak Math. J. 32 (1982) 488–494.
[25] B. Peleg, M. Yaari, On the existence of consistent course of action when tastes are changing, Rev. Econ. Stud. 40 (1973) 391–401.

[26] E. Phelps, R. Pollak, On second-best national saving and game-equilibrium growth, Rev. Econ. Stud. 35 (1968) 185–199.
[28] D. Read, Is time-discounting hyperbolic or subadditive?, J. Risk Uncertainty 23 (2001) 5–32.
[29] P. Roelofsma, D. Read, Intransitive intertemporal choice, J. Behav. Dec. Making 13 (2000) 161–177.
[30] A. Rubinstein, Is it economics and psychology?: the case of hyperbolic discounting, Int. Econ. Rev. 44 (2003) 1207–1216.
[31] P. Samuelson, A note on measurement of utility, Rev. Econ. Stud. 4 (1937) 154–161.
[33] R. Strotz, Myopia and consistency in dynamic utility maximization, Rev. Econ. Stud. 23 (1956) 1387–1406.
[34] K. Vind, Independent preferences, J. Math. Econ. 20 (1991) 119–135.
[35] P. Wakker, The algebraic versus topological approach to additive representations, J. Math. Psych. 32 (1988) 421–435.
[36] P. Wakker, Separating marginal utility and probabilistic risk aversion, Theory Dec. 36 (1994) 1–44.
[37] M. Zdun, On simultaneous Abel equations, Aequationes Math. 38 (1989) 163–177.